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AND

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

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ERRATA.

Vol. IX., p. 200.—In column 9, at age 55: 2·594 should be 2·294. (See *ante* p. 128, fifth line from bottom.)

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Some Reasons for thinking the System of Reassurance undesirable. By BENJAMIN NEWBATT, *Esq., of the Clerical, Medical, and General Life Assurance Society.*

[Read before the Institute, 25th April, 1864.]

IT has become the practice of late years for Assurance Companies, when applied to for a policy of very large amount, to issue such an one as is required and to reduce the risk within the limits they have prescribed for themselves by effecting reassurances of portions of it with other Companies. The only alternative proceeding would of course be to issue a policy for the amount to which the Company limits itself, and to leave the applicant to effect the remainder of his proposed assurance where he may. Mr. Newbatt's paper, which, as having reference to a matter of official practice merely, is not inserted here, advocates the latter mode of procedure in preference to the former, and adduces a variety of arguments in support of that view.

After some preliminary observations as to the origin of the contrary practice, he proceeds to state that it may serve to make clearer the views he entertains to look at the system under three different aspects, as it affects (1st) the Office employing it, (2nd) the Office granting the reassurance, (3rd) the interests of life assurance in general. Under the first head, he states his objections to any Company accepting a larger risk than it intends to retain;

urging that, in every such case, it in effect guarantees, without any corresponding benefit, the engagements of all the other Offices amongst which the risk has to be divided; that it may have to pay a higher extra premium than it can itself charge, or to accept a less surrender value than it is called upon to give; and that it is scarcely possible so to adjust the bonuses to be declared by the several Companies concerned in the risk as that the Company making the assurance shall escape without loss on that score. Under the second head the writer expresses his opinion that business of the description in question is transacted in a much less careful way than the ordinary business is; that it was formerly the practice, in dealing with large risks, for certain of the Companies to which proposals had been made to share the duty of investigating the case, and if they did not always make a joint report, they at least came always to a clear understanding with one another as to its merits; whereas, under the existing system, the Company accepting the gross sum collects its own evidence, which may or may not be satisfactory to others, and on that evidence asks, as a matter of course, for the acceptance of its colleagues, which is, perhaps, reluctantly given or reluctantly withheld. The evils arising from this change are particularly manifest, as the writer thinks, in a case to which he alludes, and in which, had independent inquiries been made, facts would have been discovered whilst discovery was of use. As regards the effect of the system of reassurance on the interests of life assurance generally, Mr. Newbatt observes, that if the success and prosperity of one Office tends to the good of all, the converse proposition must be admitted, viz., that none can fail to participate in some measure in the downfall or discredit of any one; and, therefore, if he has shown that in the working of the system the Company practising it, or the Company on which it is practised, suffers harm, he has at least suggested ground for looking well to its true bearings. In so far as the system tends to a monopoly of business in favour of any Company, it is opposed to the common good, and must in the end operate injuriously towards the monopolist. The very benefit which might most be insisted on is therefore a source of danger. After some further observations to the like effect, the writer concludes by saying that he has the satisfaction of knowing that the considerations he has brought forward will not be suffered to pass unheeded, and that he hopes therefore they may germinate in a change of practice.

Some Considerations on the Government Life Annuities and Life Assurances Bill. By MARCUS N. ADLER, M.A., *Fellow of the Institute of Actuaries and of the Statistical Society.*

[Read before the Institute of Actuaries, 25th April, 1864, and ordered by the Council to be printed.]

THE measure by which the Chancellor of the Exchequer proposes to extend the benefits of life assurance and annuities to the nation at large has now been before the public some time. Few Bills have of late attracted so much attention, and have been so earnestly discussed by all classes, as this. But beyond what was elicited at an interview that took place between the Chancellor of the Exchequer and the Actuaries of several Offices, and some passing remarks on the subject made at the last meeting of the Institute of Actuaries, when we had the advantage of hearing Mr. Samuel Brown's excellent paper on Friendly Societies, the public have as yet had no opportunity of hearing the opinions of those, who after all are best able to judge. It now appears, that the Chancellor of the Exchequer is opposed to a general inquiry and the calling for "persons, papers and records" by the Select Committee, to whom the Bill has been referred, and it therefore becomes all the more desirable that the merits of the proposition of the Chancellor should be calmly and impartially discussed by the members of this Institute.

Perhaps it may be considered boldness, that one so young as myself should venture to offer remarks on a measure so important as the one under consideration; but I am filled with the sincere desire of contributing, however slightly, to the passing into law of a Bill, that may be productive of incalculable benefits to the poorer and working classes, and I shall be satisfied if the comments which my paper may call forth from the members throw some new light on this subject.

The idea of providing for a sum payable at death is much older than is generally imagined. A recent discovery places us in a position to date it as far back as the time of the Emperor Hadrian. Mr. Kenrick, in his work on Roman Sepulchres, gives us a translation of an inscription on a marble slab found at Lanuvium, an ancient town in Latium, distant about 19 miles from Rome. It contains the regulations of a Club, which had for its ostensible object the worship of Diana and Antinous; but in reality it was instituted in order to provide, by monthly payments of 3 asses (about 2*d.*), a sum of 300 sesterces (about £2. 5*s.*) at death, to cover the expenses of burial. It may interest the advocates of

convivial gatherings among Friendly Societies to know, that the entrance fee of the new members consisted, besides a payment of 100 sesterces (about 15*s*.), in presenting the Society with an amphora of wine, about 6 gallons of our own measure. Here we have the features of a regular Friendly Society placed before us in a very marked manner.

Again, the idea of a Government taking in hand the granting of annuities and assurances is not of a character so novel or unheard-of as some are disposed to think. Two hundred years ago the enlightened and renowned Grand Pensionary of Holland, De Wit, counselled and prevailed upon the States of Holland to negotiate funds by life annuities, considering that to be the best mode of investment for private families.

In this country, as far back as 1773, a Bill passed the House of Commons, on the motion of Mr. Dowdeswell, to enable the purchase of deferred annuities, or provisions for old age, to be secured out of the poor rates of the parish to which the purchasers belonged, in case the funds subscribed should prove insufficient to pay the annuities. This Bill, notwithstanding that it had the support of such men as Baron Maseres, Edmund Burke, Benjamin Franklin, &c., was thrown out by the Lords. In 1807, Mr. Whitbread introduced a Bill for the establishment of Post Office Savings Banks, and of a Poors Insurance Fund in connection with the Post Office. It may be interesting to know some of the remarks made by Mr. Whitbread when introducing the Poor-Laws Bill:—

"I would propose the establishment of one great national institution, of the nature of a Bank, for the use and advantage of the labouring classes alone; that it should be placed in the metropolis, and be under the control and management of proper persons, to be appointed; that every man who shall be certified by one justice, to his own knowledge or on proof, to subsist principally or alone by the wages of their labour, shall be at liberty to remit to the accountant of the Poors Fund (as I designate it), in notes of cash, any sum from 20*s*. upwards, but not exceeding £20 in any one year, nor more in the whole than £200; that once in every week the remittances of the preceding week be laid out in the 3 per Cent. Consolidated Bank Annuities, or in some other of the Government Stocks, in the name of Commissioners to be appointed; to avoid all minute payments, no dividend to be remitted till it shall amount to 10*s*.; and that all fractional sums under 10*s*. be from time to time reinvested, in order to be rendered productive towards the expenses of the Office.

"The plan will be more amply detailed in the Bill itself, and such regulations are provided as will, with the intervention of the Post Office, give ample facilities to its execution. Gentlemen need not be told that the perfection attained in the management of that great machine is such, as to give the most easy and rapid means of communication with the metropolis,

much greater indeed than usually subsists between the remote parts of any country and its capital town.

"Sir, the advantage of such a plan as that which I have just sketched out, would be very much increased if, in addition, an opportunity were given to those who might wish, by an annual payment up to a given age, to purchase an annuity for the remainder of their lives, or to insure the payment of a gross sum to their families upon their death, or upon any of those calculable events, which are the usual objects of insurance.

"There are Offices in which the higher and middle classes may, by proportional annual payments, make a provision for themselves or families; but the lowest of the requisite payments are above the scale of the labourer, to whom such a provision is still more necessary.

"I would therefore propose, that at the same place there should be established, under the same direction, an Insurance Office for the poor. That tables should be calculated for the assurance, in consideration of annual payments, of gross sums upon the death of the assured, of an annuity for the remainder of a life after a given age, or of an annuity to a wife surviving a husband, or of payments upon a child's attaining a certain age. No annual payment to be less than 10*s*., or more than £5. That the calculations be at such rates of interest and probabilities of the duration of life, as to be likely to give such an advantage only to the insurers as would cover the expense of the establishment. That the receipts be invested in Stock. That no insurance be made upon any life without the testimony upon oath of a medical man, that such person is in good health, nor without proof on oath of the age, and the certificate of a justice that he is satisfied of the facts. On proof of fraud or misrepresentation the insurance to be forfeited. All dividends and annual payments should be wholly exempt from the property tax."*

Mr. Whitbread, in continuation, made sundry excellent suggestions with regard to the proposed Government Assurance Office, to which it would be well if the Chancellor of the Exchequer, in the measure now under consideration, were to pay regard.

The Bill was, however, rejected by Parliament. Let us hope that, after an experience of above fifty years, the present measure will meet with a better fate, and that as one part of Mr. Whitbread's plan has been realised in the establishment of Post Office Savings Banks, a Poors Insurance Fund will meet with like success.

Dr. Farr, to whom we are indebted for so many valuable contributions to the theory and practice of life assurance, has strongly advocated a plan similar to the one under consideration.

In his letter to the Registrar-General, appended to the 12th Registrar-General's Report, he says, "It is a signal defect in the existing system of insurance that in it, as in the old system of banking, the most numerous and not the least valuable, or collectively the least productive, class of the community, enjoys few

* *Hansard's Debates*, 19th February, 1807.

of its advantages; nor do I see how the defect can be adequately supplied, unless the Legislature authorise the Government to insure the lives of the people up to a certain amount, as well as to sell them deferred annuities."

Dr. Farr at the same time very happily comments upon our Poor Law, which, though not perfect, does honour to the sagacity and kindness of our Legislature. He considers the English Poor Law nothing less than an insurance of the life of every man, woman, and child in England against the danger of death by starvation. The Poor Law might be extended so as to make the relief in destitution and distress bear some proportion to the ratepayer's previous contributions, and thus put the whole population of this country on the footing of a great Friendly Society, in which the higher classes are honorary members, who, in the event of their being reduced to poverty, might be relieved on a scale commensurate with their previous contributions. At all events any plan which is calculated to aid the workman in old age, or his family after his death, may be looked upon as an improvement of the Poor Law, and may therefore well be taken up by Government.

Mr. Tidd Pratt has advocated the establishment of parochial Friendly Societies, and the authorisation of the Trustees of the poor, in the various parishes of the United Kingdom, to defray the expenses of the formation and management of one soundly-constituted Friendly Society in their districts, provided they have a right of supervision or participation in the management.

The Marquis of Lansdowne would have ventured even further. In 1861 he introduced a Bill—a permissive one—empowering a parish, or union of parishes, to vote from the poor rates one-fourth part of the yearly contributions of the members. This Bill is founded upon one amended in Committee in the year 1818, the preamble of which propounded, that with a view to the reduction of the poor rate, and to the gradual introduction of a better feeling among the people, it was desirable, that special encouragement and facility should be afforded to meritorious and industrious persons for rescuing themselves from the necessity of an appeal to parochial relief.

Thus repeatedly and emphatically has the interference of Government, or of the local authorities, been advocated.

The present measure, which was so ably introduced by the Chancellor of the Exchequer, has two objects in view:—

First, that annuities authorised to be granted under 16 & 17 Vict., chap. 45, may be purchased on payment of smaller instalments and at shorter periods than were fixed under the said Act.

Few, I think, will be disposed to dispute the propriety and fairness of this proposition. It is only necessary to mention, that, at the present time, only 6,500 such annuities are in force under the Savings Bank Act, securing in the aggregate, annuities for less than £140,000 a year, to show that these annuities have hitherto been neither popular nor have been extensively adopted.

I question, whether, even with the facilities which the Chancellor of the Exchequer now offers, life annuities, in their present form, will become more favourite investments for working men; and yet what could be more desirable, than that the industrial classes should make provision for their old age from their savings during the years they enjoy health and vigour? According to the rates now charged at the National Debt Office, a man aged 20, to secure for himself a deferred annuity of £13 a year, or 5*s.* a week, to commence on his attaining the age of 65, would have to pay 13*s.* a year on the non-returnable, and £1. 2*s.* 9*d.* on the returnable scale. According to the former, if illness or any other cause should prevent him from making one of the payments at the proper time, he forfeits both the annuity and the premiums paid. According to the latter scale, it is true he may have the premiums paid, returned at any time, but if he should once omit paying the premium at the proper time, he is deprived of the annuity that is to support him in old age.

Dr. Farr proposes, in the 12th Registrar-General's Report, a plan according to which a separate annuity can be purchased for every premium, which may be discontinued at any time. Thus a man aged 20, can, for £1, secure a deferred annuity of 17*s.* 5*d.* on his attaining the age 65; if he repeat this payment the following year, he secures altogether an annuity of £1. 14*s.* 2*d.*; for 10 years premiums of £1 a year he can secure an annuity of £7. 9*s.* 8*d.*; for 21 years premiums he can secure an allowance of 5*s.* a week; and if he continues to pay till the year 65, he can provide himself with an allowance of above 7*s.* a week, the premiums paid being returnable at any time he pleases, provided he be in good health; whereas, according to the existing rates, an allowance of but 5*s.* a week can be secured, as will be seen from the Government tables, on an annual payment of £1. 2*s.* 9*d.* This is a startling difference; and, although in the latter case the premiums are returnable, even if the state of health of the life is impaired, yet, if we remember that, according to Dr. Farr's plan, the purchaser is not bound to pay the same amount each time, but may deposit whatever sum he can spare, monthly or even more frequently, without giving

more trouble than he would cause on paying a deposit to a Savings Bank, except the necessity of referring to a table to ascertain what amount of annuity shall be placed to his credit in respect of the payment made, this method will be found very advantageous.

True no charge on the premiums for expenses of management has been made, but that would little interfere with the working of the measure; and it might well be argued, that as the present Government annuity rates do not appear to be loaded, though calculated at $3\frac{1}{4}$ per cent., annuities on the method pointed out could be granted if computed at 3 per cent., without subjecting the rates to any addition. Besides, as the grant of life annuities tends to the reduction of the National Debt, the State may well bear the expense of the management of the business. I may add, that there is nothing in the wording of the proposed Bill, which could prevent the scheme I have just pointed out from being adopted.

The other object the Chancellor of the Exchequer wishes to attain is, to make life assurance more accessible to the working classes, by removing the restriction under which persons buying a life assurance were obliged at the same time to purchase a deferred annuity. The existing law is evidently unreasonable, because the man, who wishes to provide for his family after his death, need not at the same time be anxious to provide for his old age.

Yet the Chancellor's proposal has met with great opposition. Loud was the clamour raised against it, in the first instance, from many different quarters. The agitation seems now to be gradually subsiding; and there cannot be a more conclusive proof of the increasing popularity of the measure than the fact, that the whole body of Life Assurance Companies in Scotland has lately sent a statement to the Chancellor of the Exchequer expressing its general approval of the Bill.

Let us examine the objections that have been raised against the Chancellor of the Exchequer's proposition.

One chief argument advanced against Government granting assurances, is, that the State has no right to interfere in such matters, life assurance being a regular business, like banking, trading, baking and brewing, or any industrial occupation.

In reply to which we need only call to mind the remarks previously cited, made by Dr. Farr and other eminent philanthropists, showing how the State is directly benefited by the poor man's provision for his wife and children; to what extent the entire nation reaps the advantage, if inducements are held out to the poor to become

thrifty and industrious. By granting assurances the State does hold out these inducements. In the words of Professor De Morgan, who 30 years ago advocated the establishment of one large central Assurance Office, the expense of which he even suggested should be borne for a few years by the public purse, "The Act which should establish this universal Friendly Society would, in two generations, become the real Poor Law." *

But it is alleged, that, by founding a Government Insurance Society, the rights of the people are attacked, and their freedom infringed upon. We answer, their freedom would in a similar manner have been interfered with by the establishment of the Government Annuity Office, or of the Post-Office Savings Banks. There is no compulsion in the matter; people may choose for themselves whether they will invest therein or not. Even John Stuart Mill,† whilst strongly advocating the "laisser faire" system, admits that, when a Government provides means for attaining a certain end, leaving individuals free to avail themselves of different means, if these be in their opinion preferable, there is no infringement of liberty, no irksome or degrading restraint.

It is asserted that the people will be taught thereby to rely more upon Government than upon themselves. It is quite true that Government will, by means of this measure, do away to some degree with the necessity of operatives, who hardly know their alphabet, framing tables of rates of mortality; or, in the words of Dr. Farr, "playing with the artificed tools of actuaries." In that respect it will be better if they trust to others rather than to themselves. But does the Government subsidize them, pay their premiums for them, or allow the people to relax in their exertions to gain a respectable living and to lay by what they can? On the contrary, by showing the operatives where they can invest their premiums, even in very small amounts, with absolute safety, habits of thrift and feelings of independence are fostered; they will be made to rely more upon themselves and less upon Government by securing themselves in their old age, and their families after their death, against the necessity of having to resort to the parish for relief. It would also be an advantage to the State to put the working man in possession of a security not altogether tangible, but which, by means of the law, becomes possessed of value. If, also, the support of his old age, and that of his family after his death, becomes connected with his past savings, the labouring man

* *Essay on Probabilities*, p. 298.

† *Principles of Political Economy*, book v., chap. ix.

becomes clearly interested in the upholding of the existing state of things.

Another argument against the Bill was, that by placing insurance in the hands of Government, there would be a great increase in the number of Government officials; by the creation of which, according to J. S. Mill, notwithstanding all the liberty of the press, or the popular constitution of the Legislature, our country would become free only in name. Let us look at the state of the case. The Government proposes to avail itself of a machinery already in existence. The Post-Office officials are to act the part of agents, and the Poor-Law medical officers the part of medical examiners. There will, at most, be a necessity for appointing some commissioners for the control of the establishment; an actuary, an accountant, and a staff of clerks to conduct the business at the head office; perhaps also some travelling inspectors to supervise the different agencies.

Again, it is asserted that the cost of such an establishment, if carried on by Government, will be large; and it is unjust that the public should bear this expense for the benefit of the few, that will avail themselves thereof. It has been seen that such an Office will, though it may work upon a large area, yet not involve heavy expense; a small increase of salary might be sufficient for the remuneration of the Post-Office employées. It has also been seen that the State, or its representatives, the Poor-Law Commissioners, are directly interested in the extension of life assurance among the working classes, so that the aid of those in their employ, the Poor-Law medical officers, may well be claimed, nor need the pay for their services be large. But whatever the expenses may be, they will not fall upon the public purse; the mode proposed to be adopted will be to put an adequate *loading* upon the premiums, which will meet all the expenses.

But, it is argued, it is not fair for Government to enter into competition with Insurance and other Societies, and make use of the resources of the nation to enable it to undersell public Companies. If Government does enter into such a course, it will, it is contended, establish for itself a monopoly. The like outcries have been raised on previous similar occasions. With what reason let us see.

When Post-Office Orders were about to be introduced, those interested in Banks complained and expressed their conviction that Government would not be satisfied with issuing money orders for small amounts, but would gradually take the banking business in their own hands, an attempt which would injure all private

concerns. But what is the real state of the case? Government has not exceeded the maximum of £10 for each money order, and banking business has become developed to the remarkable extent we witness at the present day.

Again, when but lately the Post-Office Savings Banks were opened, many foretold the ruin of the old Savings Banks. But what is the result? At the end of 1863 the capital of the Savings Banks was £41,258,000, being only £20,000 less than the capital at the end of 1860, before the Post-Office Savings Banks had been established. Now, if we remember that in consequence of the cotton famine the amount withdrawn in the year 1861-2 exceeded the average by above half a million pounds, we shall find that, allowing for this extraordinary efflux, the deposits with the old Savings Banks have actually increased—increased even by a considerable amount. The Post-Office Savings Banks themselves cannot but be considered an eminent success. After having been but two years in existence, nearly three million and a half pounds stood invested therein last year.

A similar result, I believe, will happen with Assurance Offices. The Government will, no doubt, obtain a considerable amount of business; but the field of operations is so large, that neither the deserving Friendly and Industrial Societies, much less the class Insurance Companies, will suffer thereby. Let us look at the advantages the latter possess over a Government establishment. In the first place, as the funds of the Government Assurance Office are only to be invested in Government securities, they cannot be considered as realising much more than 3, at the utmost 3½, per cent., whereas the old established Insurance Companies make between 4 and 5 per cent. An addition sufficient to cover all the expenses of management will also have to be made to the net premiums. Hence the rates the Government will have to adopt will be higher than the non-participating rates of most of the other Offices. We cannot expect them to be higher than the participating rates of those Offices, because these make a periodical return to the assured in the shape of reduction of the premiums, or bonuses, which the Government does not propose to do. Government will not, then, undersell the public Companies.

Much less will it establish a monopoly for itself. It does not propose to take assurances for amounts larger than £100; possibly, if Mr. Sheridan's motion passes, the limit will be £50. Now, this is dealing with the worst class of business; that branch, in fact, which, it is justly said, is least remunerative.

The remarks made by the chairman of the Economic Life Assurance Society, at their last general meeting, are so appropriate, that I cannot refrain from quoting them. He says:—

"The Chancellor of the Exchequer is going into a business we do not at all covet. We are quite willing to leave him to carry on his own business; for no doubt it is the duty of a paternal Government to look after the industrial classes, and we propose to let him do so without cavilling at such a benevolent intention. I can point out, without much difficulty, the way in which we looked at it. We assure as low as £100, but we do not like such small policies. The larger the policy, the better it is for the Society generally; and for this reason, that our premiums are calculated with a certain margin for various little chance deviations in the calculations as to the exact risk of life, and also to pay the expenses of the Office, and to put by such profit as we can. Supposing this margin was 25 per cent.; if you assure a life at an age at which the premium would be £4 for £100, 25 per cent., or £1 of that would go to the Office expenses and to accumulations of profit;* but if you assure a person for £1,000 at the same age, you get £10 for the same purpose, and the expense to the Office is not 6*d.* more. So that the smaller the amount of the policies, the greater is the proportional expense. We find, from experience, that each policy costs us nearly £1 per annum. No doubt the Chancellor of the Exchequer is up to all this; and, therefore, he is obliged to state candidly, that in order to pay his way he must charge higher rates of premium than an ordinary Assurance Office would do."

Government also proposes not to take any assurances on lives under 16 years.

It is clear, then, that under the present conditions there is no cause for fear of the Government creating a monopoly. But what guarantee, it is asked, have we, that Government will stop at this maximum of £100? "It is now getting the thin edge of the wedge in, and will usurp, at some future time, the whole of the insurance business." We say, in reply, that such a course would completely set at naught the very object for which the present Bill has been proposed. It has not in view the reduction of the National Debt by the conversion of a permanent into a temporary charge, or any similar aim. Its only avowed and implied object is to offer facilities for the increase and extension of frugal habits among the working classes, and those only. It is based upon the same principle as that of Post Office Savings Banks, and we have not yet heard of any attempt to increase the maximum of deposit for those establishments. Besides, it is unlikely, even if Government were empowered to grant assurances above £100,

* Properly speaking, only 16*s.* in the case where the pure premiums are loaded 25 per cent.

that it would find customers. Those who insure for larger amounts can form some opinion as to the status of different Offices, and would certainly prefer those which give large bonuses to the assured, to a Government Insurance Office which asks high rates and makes no returns.

I have heard others shift the argument, and assert that the State should shrink from taking in hand that branch of life assurance business, which is least remunerative—a business which the larger Companies hesitate to transact. The system, they say, must prove a failure, and ultimately cause great loss to the country. But the Chancellor of the Exchequer has pointed out to us a mode by which these small assurances can be taken, which, with so well disciplined a staff of officers as that of the Post-Office, cannot fail to work well, and largeness of number will effect what in the present Assurance Companies is attained by the greatness of the amount. Assurances granted by Government in large numbers for small amounts would also possess that advantage, that there is much less likelihood of departures from the mean average. The risk of a high average amount assured becoming claimable in certain years is thereby materially lessened. This fact will render it almost unnecessary for the Government to set apart any guarantee fund to meet excessive claims at any time. If every facility be given to the working classes—possibly even by allowing Friendly Societies and Savings Banks to act the part of intermediate agents between the working classes and the Government—there is no reason why the plan should prove a failure. Instead of its proving a loss to the country, as its opponents prophesy, it may even prove a source of profit; for no doubt, if sufficiently high premiums are charged, a profit must be realised, to part of which the State has a right, guaranteeing as it does the obligations of the Assurance Office.

But, it is argued again, why take from Friendly Societies the most profitable part of their business, and leave them the least profitable portion? Why deal a blow at institutions which evidently suit the views and the taste of the working man, and which are calculated to uphold that eminently English feeling for local self-government by which the nation becomes best fitted for the enjoyment of political rights?

To the first portion of this argument we reply, in the words of an able writer in *All the Year Round,* "The Government will take away from Friendly Societies not that part of their business which is necessarily the most profitable, but that part of their business which enables them to conceal their insolvency for the longest

period, and which for the longest time facilitates a lavish and wasteful expenditure."

I would not wish you to understand that I am at all opposed to, or would animadvert upon, the Friendly Societies; but I cannot help thinking that something should be done for their maintenance. The plan at present proposed, viz., to establish a Central Committee, independent of Government—similar to the Friendly Societies Institute, which has lately been dissolved—and which committee would discharge the functions of a consulting actuary and otherwise advise the Friendly Societies, will not, I think, answer the object in view. Take the case of the Royal Liver Friendly Society, which is so indefatigable in its opposition to this Bill, and whose different branches are every day petitioning Parliament against the measure. An eminent actuary certifies that the rates are adequate for providing the benefits—which no doubt they are; but he takes care to add, "the transactions connected with such a branch of provision being presumed to involve very trifling expense." Now what is the real nature of the case? Mr. Tidd Pratt, in his Report of 1863, on Friendly Societies, states that the total expenditure of the Society amounted to 40 per cent. of its income; and in previous years it was even higher. That was certainly not a trifling expense. Yet I am far from saying that this and similar Institutions are in an insolvent condition. I think there is a good deal of truth in what Mr. Sprague advanced at the last meeting, that in consequence of the premiums charged by Industrial and Friendly Societies being very heavily loaded, often as much as 60 or 70 per cent., several years may elapse before the net premium that would then be charged amounts to the premium levied originally. Accordingly there may be Societies of 5 or 6 years' standing, having but a trifling reserve, that may yet be in a solvent condition. But what I urge is, that it is of no use for actuaries, or any counselling body which cannot enforce its authority, to calculate tables of rates, to make certain suggestions, and lay down principles for the guidance of Friendly Societies, if these Societies depart therefrom and follow their own opinions.

I conceive, however, that it would be less objectionable for Friendly Societies to continue to grant sick allowances, especially if these sickness payments are only allowed to be made up to age 60 or 65. In the words of our excellent President, Mr. Jellicoe, "Sickness Clubs do not necessarily involve any lengthened series of

accounts, or any prolonged maintenance of funds. There is not in them that lamentable and most painful character which pertains to Assurance Clubs, or the Deferred Annuity Clubs, to which men may subscribe for years and years, and when the time for the expected benefit comes a total failure ensues. In Sickness Clubs there need be nothing of the sort. It is quite competent for a number of persons to subscribe together for the sickness of this year, next year, or the third year, and having paid all demands upon the Club, to settle its affairs and begin *de novo.*"

If, at the same time, these Friendly Societies acted the part of intermediate agent between the working man and the Government Office for the effecting of assurances and deferred annuities, the Societies would still continue to exercise a beneficial influence. The members could continue to enjoy their meetings, and discuss amongst each other the common objects of their Society. It would not add much to the duties of the collector of a Friendly Society, who, in order to collect the weekly payments to cover against sickness, &c., has to go his round from house to house, if, for a trifling commission (not for 25 per cent. as hitherto), he were to collect weekly the premiums of those who, through the agency of the Friendly Society, assure with the Government Office. There would be no chance of the Society spending the funds so collected, as the premiums would have to be regularly paid, say every month, by the Society to the Government Office, which might then issue, as is even now the case with the Post-Office Banks, an official receipt for the premium direct to the assured. I believe a similar course is already being pursued in the case of the deferred annuities granted by the National Debt Office; the Friendly Society or the Savings Bank negotiating, on behalf of their customers, with the National Debt Office, for the purchase of an immediate or deferred annuity. But even if this arrangement be not adopted, the establishment of a Government Assurance Office will doubtless cause the habit of assurance to become much more general, not only amongst the operatives and others of small means, but also amongst the middle and higher classes, so that neither the Friendly Societies nor Insurance Companies will, after a few years, have more cause to complain of Government competition, than, as we have shown, the Banking Companies have on account of the issue of Post-Office Orders, or Savings Banks on account of the Post-Office Banks. Besides, Government does not propose to withdraw from the numerous Friendly Societies the high rate it allows on their

investments,* a rate much higher than the State itself will, generally speaking, be able to realise on its investments. These Friendly Societies, if only properly managed, may successfully compete with Government.

Having thus referred to the case of Friendly Societies, and shown that they will not be injured, but rather benefited by the Government measure, I will but glance at one or two other objections which have been raised.

It is said that the careful selection of lives is likely to be carried out much better by directors of Companies and private establishments, who are personally interested in the matter, than by Government officers who are not. In reply, it must be remembered, that most of the business of life assurance is transacted by agents, some of whom do not much consider the eligibility of the life for assurance, but, apart even from considerations of commission, are desirous that their friends should be accepted. The directors have only in town cases, and then not always, the opportunity of seeing the life; in all other cases they are solely guided by the opinion of the agent, the report of the medical examiner, and a letter from a friend of the assured, who not rarely writes what the agent or the life proposed dictates to him. It appears, then, that the steps at present taken by Companies, to form an opinion as to the eligibility of a life are not always satisfactory. On the other hand, according to the Government plan, neither the Post-Office employées nor the other officials would be anxious to pass proposals merely for the sake of getting new business; and there is no reason in the world, why they should not be as conscientious and careful as those holding other appointments. In a large number of cases, postmasters are even at present very efficient agents to respectable Assurance Companies, and they find sufficient time to transact the business in connection with their agency.

Again, the effects of selection of lives have been very much exaggerated. The result of an investigation on the subject by Mr. Farren is, that the rates of mortality of persons insured would not particularly differ from those prevailing among the male population at large, taken indiscriminately without regard to health. Again, Mr. Samuel Brown says, "The general tendency of selection

* From a recent Parliamentary Return it appears that there are 99 old Societies, with a capital of £593,523, enjoying £4. 11*s.* 3*d.* per cent. per annum interest; and 352 Societies, with a capital of £1,256,308, receiving £3. 16*s.* 0½*d.* per cent. per annum interest. Only 219 Societies, with a capital of but £190,893, receive £3. 0*s.* 10*d.* per cent. per annum interest.

cannot be mistaken, and it is universally found, that if the mortality in the first few years be less, it increases very rapidly after a short period." The experience of Mr. W. Morgan, Dr. Farr, Mr. Higham, and other eminent actuaries, tends to the same result. Mr. Scratchley goes even so far as to urge that the field of assurance should no longer be limited; that attempts should be made for the general assurance of lives, however apparently they may have departed from the assumed standard of average good health, a suggestion which several Offices are acting upon. Though the effects of selection have thus been exaggerated, it does not follow that applicants should be indiscriminately admitted to the benefits of life assurance. I do not even consider it sufficient for the Office to satisfy itself on the three points mentioned by the Chancellor of the Exchequer, viz., the age, the employment, and habits of the life proposed for assurance. Government would soon find such a course absolutely ruinous, for all those afflicted with serious maladies, and those who feel death impending, would crowd to the Government Office and assure their lives without delay. I therefore think that applicants should not be admitted to life assurance without due care being exercised, and that a preliminary medical examination should be a *sine qua non.*

Having answered as many of the objections raised against the Bill as I have heard advanced, let me now show some of the positive advantages that the measure holds out. The administration of the Government Assurance Office will, no doubt, be as perfect as the united endeavours and abilities of those we consider the most competent in the country for that purpose—those, in fact, to whom we entrust the management of our finances—are able to render it. No private Company possesses this advantage.

The greatness of the number of the assurances that will be granted will prevent an undue departure from the average.

The aggregate expense of management will be much less than if such assurances were granted by a large number of small Companies, all competing with each other to obtain business.

In the event of an assured changing his place of abode, he would not risk forfeiting his assurance, as those who belong to Friendly Societies are liable to, but the assurance would merely have to be transferred from one postmaster to another. The value of this privilege will be at once perceived, if we remember that many of the depositors with the Post-Office Savings Banks have availed themselves thereof.

We now come to the question, as to what table of mortality

should be adopted by the Government Assurance Office. It becomes necessary to consider what class of society is most likely to assure with the Government. I think it will be found, that the real working men of our country do not care to provide their families with any considerable sum to be received after death; the utmost they aspire to, is to provide for a decent burial and a small sum of money, which is sure "to come in handy" on those occasions. Besides, the Chancellor of the Exchequer proposes not to receive any sum in respect of instalments or premiums of a less amount than 2*s*. The ordinary operative will hardly be able to spare this amount from his wages at any one time; he will therefore continue to assure with the Friendly Societies, till Government offers proper facilities. But a superior class of the population will, I think, readily avail itself of the benefits the Government holds out—tradesmen, artizans, clerks and servants.

Now, the tables of mortality deduced from the returns of Friendly Societies, and framed by Mr. A. G. Finlaison, will not altogether apply. Though these useful tables are quite correct, and are borne out by Mr. Ratcliffe's and Mr. Neison's observations of the same class of Society, they would give far too favourable a view of the mortality likely to be experienced by those who will avail themselves of the Government terms.*

The Northampton table, with its acknowledged shortcomings, as well as the Carlisle table, with its imperfect graduation, the limited extent of the observations on which it is based, and its giving only the mortality of one single town, can at once be dismissed as undesirable to adopt.

The tables of mortality deduced from the experience of the Equitable, or from that of the seventeen Offices, ought not to be used, because they give the mortality of the middle and higher classes only.

The Government tables constructed by Mr. John Finlaison are based upon the recorded ages and deaths of 22,000 Government annuitants and tontine nominees in England and Ireland. These tables, it must be admitted, representing as they do the mortality of

* *Vide* Report and Tables by the Actuary of the National Debt Office, on the subject of sickness and mortality among members of Friendly Societies, presented to Parliament 12th August, 1853; and "Observations upon the sickness and mortality experienced in Friendly Societies, by Henry Tompkins" (*Assurance Magazine*, vol. v., p. 13). *Primâ facie* it would appear, that the circumstances in which the working population of this country is placed are decidedly opposed to a prolonged duration of life, but observations upon Friendly Societies leave no doubt as to the fact, that the value of life of the labouring classes is much superior to that of the general community.

the richer classes—of those who do not know what manual labour is, and more especially of annuitants who apply only for an annuity if they know by themselves that they will long enjoy it—and after all they are the best judges—these tables, I think, will not be a criterion of the mortality of those about to patronize the Government Assurance Office.

I imagine, then, that the English Life Tables, more especially No. 2, which are based upon the recorded ages of above 15 millions living persons and nearly $2\frac{1}{2}$ million deaths, and in connection with which every possible circumstance seems to have been taken into consideration, will be the table that ought to guide the Government Assurance Office in its calculations. I think we may soon look forward to the publication of a third English Life Table, which will, no doubt, possess advantages superior even to those presented by the two former, and to which of course the preference would have to be given.

In the English Life Tables, there are separate tables of mortality for males and females. The distinction of male and female life is of importance in the granting of annuities, because a large number of annuitants are of the female sex, who, it is a well-established fact, live longer than the male sex. But with regard to life assurance, it would be keeping on the safe side to grant assurances on the lives of females on the same terms as to males.

With respect to the rate of interest at which the tables ought to be calculated, it must be remembered that the funds will all have to be invested in the Government Securities, such as Consols. Now, it is an undoubted fact, that the rate of interest has been slowly falling for centuries, and that, as was the case in Holland last century, the rate may descend so low, that Government can borrow at 2 per cent., and others at 3 per cent.; yet if we look at the price of securities for the last 30 years, the average price of Consols will be found to be about 92, and the corresponding rate of interest above $3\frac{1}{4}$ per cent.; and to judge from the state of the money market at the present time, and the causes that have brought the present dearness of money about, it is not unlikely that for the future the average rate may be still higher. Under these circumstances, provided a sufficient loading be placed upon the premiums, they can safely be calculated at 3 per cent. If Government should, at some future time, reduce the interest, and their assurance establishment suffer thereby, the latter might justly

claim indemnification; the more so, if part of the profits are to be paid into the public exchequer.

As to the question, what loading should be put upon the net premiums, we must remember that, even where there is no necessity to lay the foundation for a bonus fund, yet provision must be made (1) for the cost of management, (2) to form a contingent fund to meet excess of mortality or loss through investments and reduction of the interest. We have already shown that the guarantee fund need not be large. Respecting the probable expenses of management, it is difficult to form any correct idea. The expenses of the Post Office Savings Banks for the year ending 31st Dec., 1863, were £25,400. 18*s*. 6*d*., exclusive of the allowances to postmasters, letter receivers, &c., on nearly three million and a half sterling deposits; but this will hardly be a criterion as to the probable expenses of the Government Assurance Office. We have heard of Societies whose expenditure amounted to 295 per cent. on the premiums; and, on the other hand, there are offices whose expenses are but 4 per cent. of their receipts. It is Professor De Morgan's opinion* that the expenses of carrying on an Insurance Office, though they vary somewhat with the amount of business, yet do not by any means increase as fast. In the first year of its existence it would not be surprising if all the premiums paid were swallowed up by house-rent, salaries, &c.; while, in process of time, increase of business might reduce such expenditure to 2 per cent. upon the yearly premiums. True, according to the plan before us, there would be a great saving effected in the item of commission for procuring assurances, which, of course, the Government would not have to pay; yet they will have to make some remuneration to the Post-Office employées, as well as to the medical officers, proportional to the business that passes through their hands, which remuneration, however, will be but trifling compared with the high commission allowed to the agents of Industrial and Friendly Societies. The late Mr. Whitbread, in his Bill of 1807, proposed that each postmaster should receive one penny in the pound of the money passing through his hands. If great facilities were to be offered to the working classes, in collecting the premiums at their homes, monthly or weekly, the remuneration would, of course, have to be more liberal.

I subjoin a specimen Table of Rates, both net and with a

* *Essay on Probabilities*, p. 263.

loading of 20 per cent., according to the Male English Life Table, No. 2, and at 3 per cent. interest.*

Age.	Net Annual Premium.	Annual Premium with Loading of 20 per Cent.	Half-yearly Premium.		Quarterly Premium.		Monthly Premium.	
			Net.	With Charge of 20 per Cent.	Net.	With Charge of 20 per Cent.	Net.	With Charge of 20 per Cent.
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	s. d.	s. d.
16	1 8 8	1 14 5	0 14 6	0 17 5	0 7 4	0 8 9	2 6	3 0
20	1 11 8	1 18 0	0 16 0	0 19 3	0 8 1	0 9 8	2 9	3 3
25	1 15 9	2 2 11	0 18 1	1 1 9	0 9 1	0 10 11	3 1	3 8
30	2 0 9	2 8 11	1 0 8	1 4 9	0 10 5	0 12 6	3 6	4 3
35	2 7 0	2 16 5	1 3 10	1 8 7	0 12 0	0 14 5	4 1	4 10
40	2 14 11	3 5 11	1 7 11	1 13 6	0 14 0	0 16 10	4 9	5 8
45	3 5 1	3 18 1	1 13 1	1 19 8	0 16 8	1 0 0	5 8	6 9
50	3 18 6	4 14 2	1 19 11	2 7 11	1 0 2	1 4 2	6 10	8 2
55	4 16 11	5 16 4	2 9 5	2 19 4	1 4 11	1 9 11	8 5	10 1
50	6 2 7	7 7 1	3 2 9	3 15 3	1 11 9	1 18 1	10 8	12 10

The net rates, with a 20 per cent. loading, will be found to exceed the non-participating rates of nearly all the Insurance Offices. But if this loading be considered inadequate, it might be raised to 30 or even 40 per cent. on the pure premium; and even then, notwithstanding what has been asserted to the contrary in many quarters, the Government rates will be lower than those of most Friendly Societies or Industrial Assurance Offices, some of which load their premiums from 50 to 70 per cent. Government will, then, not only offer superior security, but even cheaper terms than the Friendly Societies. This cannot be called a public grievance, but rather a public benefit; whilst for the reasons already pointed out, Friendly Societies will by no means cease to exist, much less be ruined.

* The rates of premiums payable more than once a year were calculated from the annual premiums by the formula

$$\frac{2\pi_x}{2m-(m-1)(\pi_x+d)},$$

where π_x is the annual premium at age x, d the discount for 1 year of £1, m the number of times a year the premium is payable.

It is obtained in the following manner:—The formula for the premium for the mth portion of a year is

$$\frac{\pi_x}{m}\left\{1+\frac{\frac{m-1}{2m}}{\frac{m+1}{2m}+a_x}\right\}=\frac{\pi_x}{m}\left\{1+\frac{\frac{m-1}{2m}}{1+a_x-\frac{m-1}{2m}}\right\}=\frac{\pi_x}{m}\left\{1+\frac{\frac{m-1}{2m}}{\frac{1}{\pi_x+d}-\frac{m-1}{2m}}\right\},$$

$$=\frac{\pi_x}{m}\left\{\frac{\frac{1}{\pi_x+d}}{\frac{1}{\pi_x+d}-\frac{m-1}{2m}}\right\}=\frac{2\pi_x}{2m-(m-1)(\pi_x+d)}.$$

The manner in which tables of rates could be most advantageously placed before the public is, to charge uniform rates of premiums, say of 2*s.* a month or 6*d.* a week, or multiples of these rates, to cover different amounts assured at the different ages. If placed before him in this shape, the working man would more clearly see the advantages to be secured by life assurance.

The following table shows the

Amount that can be Assured by the Payment of a uniform Premium of 2*s. a Month or* 6*d. a Week for the whole Term of Life, according to the English Male Life Table, No.* 2, *at* 3 *per Cent., with* 20 *per Cent. Loading.*

Age	16.	20.	25.	30.	35.	40.	45.	50.	55.	60.
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.
Amount Assured	66 13 4	61 10 9	54 12 0	47 1 2	41 7 6	35 6 0	29 12 6	24 9 10	19 6 8	15 11 8

It would be desirable also not only to have a table of rates payable during the whole of life, but also tables of rates payable only to the age 60 or 65, which is about the age when the majority of the industrial classes cease to work, and consequently no longer receive wages. Provident men would then commence to enjoy their deferred annuity; but even if no provision has been made for such annuity, it would not be desirable that working men should be troubled and weighed down at that stage of their existence with making payments they can ill afford, to secure a certain sum after death to their families.

The following table shows the

Net Annual and Monthly Premium, to cease on the Life Assured attaining Age 60, *for assuring the Sum of* £100 *at Death; it is also computed at* 3 *per Cent., and according to the English Male Life Table, No.* 2.

Age.	Annual Premium.	Monthly Premium.
	£ s. d.	£ s. d.
16	1 10 11	0 2 9
20	1 14 8	0 3 1
25	2 0 2	0 3 6
30	2 7 4	0 4 2
35	2 17 4	0 5 0
40	3 12 1	0 6 4
45	4 15 11	0 8 5
50	7 2 2	0 12 5
55	13 17 2	1 4 7

Endowment assurances, according to which the amount assured becomes payable on the life attaining a certain age, say 60, or at death, should that happen previously, I do not think would suit the working or industrial classes.

The grant of endowments or provisions payable at the end of a certain number of years might, I think, well be embraced in the sphere of operations of a Government Assurance Office. By means thereof, the provident man might secure a provision for his daughter, at birth or at a later period, payable on her attaining a certain age, with a view to a marriage portion; or in favour of his son, to provide for his education, apprenticeship, or for otherwise starting him in life. The loading, in these cases, need not be so high as that for assurance premiums, and the premium should be so calculated that, in the event of the death of the nominee, or the inability of the purchaser to keep up the payments, the premiums paid could be returned.

The proposition of the Chancellor of the Exchequer for dealing with assurances in the event of their lapsing through non-payment of the premium, or if it should be desired to surrender them, has now to be considered. Mr. Gladstone is about to propose the following in Committee, "that, in case any person who has purchased or shall purchase a deferred annuity, under the provisions of the Act of the 10th year of George IV., c. 24, s. 36, or under the provisions of the Act of 16 & 17 Vict., c. 45, s. 5, or a payment to be made at death under this Act, after having paid the several instalments or premiums for a period of not less than five years, shall make default in the payments stipulated to be made according to the contract for the purchase of such deferred life annuity or payment on death, the Commissioners for the Reduction of the National Debt, at the option of the party beneficially interested in the contract, shall grant to such party a life annuity immediate or deferred, equivalent according to the tables then in force, under the said Act, to the amount of the several payments which shall have been actually made by such person, or shall contract to pay on the death of such person such a sum of money as would be assured by the sum already paid in premiums, or shall make immediate payment to such person of such sum of money as shall be fixed by the regulations authorised to be framed under the provisions of this Act, not being less than one-half of the payments actually made by such person."

I think it will easily be seen, that if these provisions should

pass into law the Government Assurance Office would not find this a self-supporting system, under the ordinary system of assurance. But the whole of this motion appears to be full of anomalies. In the first place, when speaking of deferred annuities, it is to be presumed that the Chancellor only refers to those taken out on the returnable or non-returnable scale. In the former case the money is returnable at any time: Why, then, propose to give only half the premiums back? If the non-returnable scale be referred to, such annuities would very nearly possess all the advantages of those on the returnable scale; and that man would indeed be foolish, who cares to purchase an annuity which he might, under another name, buy from 30 to 50 per cent. cheaper.

Again, if a man, after having been assured for five years, can, for the premiums he has paid, receive an equivalent annuity, this annuity will not only suffice to cover the additional premium he will have to pay to secure an assurance at his advanced age, but he can besides enjoy an annuity throughout life. After the lapse of every five years there is no reason why he should not repeat this process, and thus he will, in the course of time, have a very handsome annual allowance besides his assurance, thanks to Mr. Gladstone's liberality.

Let us take one numerical example:—The net premium at age 20, to assure the sum of £100, is £1. 11*s*. 8*d*. according to the English Life Table. At the end of five years he has paid £7. 18*s*. 4*d*. If he now likes to make default, he can buy an annuity for life of 7*s*. 6*d*. a year nearly. Now the premium for an assurance of £100 at age 25 is £1. 15*s*. 9*d*.; so that after assigning 4*s*. 1*d*. a year for the enhanced premium, he is still the gainer by an annuity of 3*s*. 5*d*. This process he can repeat every five years; and as similar absurdities could be shown in other cases, it becomes evident that it would not be prudent for the Government Office to follow a practice, which, I may add, no Insurance Office in England has been venturesome enough to embrace.

I think it will also at once be seen, that, if undue facilities are offered for surrendering a policy, people are apt to withdraw the premiums paid when money is dear, and the price of securities low; a step which would necessitate the Government Assurance Office to sell out stock at a low price, and at an eventual loss. No doubt it will be recollected what cost this entails upon the nation in the case of the Savings Banks.*

* From a return dated 31st March, 1864, it appears that the amount of money, principal and interest, due to the Trustees of Savings Banks by the Commissioners of the National Debt, on the 20th November, 1863, was £41,237,932; the value of the securities held by the Commissioners to provide this amount was £38,554,846. 6*s*. 4*d*., leaving a deficiency of £2,683,085. 13*s*. 8*d*.

Besides, it would not be judicious to offer too great facilities for withdrawing the premiums paid in respect of assurances. The payments made should be looked upon as investments for the future benefit of the family, and not as a deposit "at call," to serve merely temporary advantages.

Out of doors, there is no part of life insurance which is so generally misunderstood as that of the surrender values and the lapsing of policies. In the words of Professor De Morgan:—

"Persons having insured for their whole lives, and being afterwards desirous to discontinue, are surprised to find that they cannot get for their policies even as much as the amount of their premiums, to say nothing of interest. Each of them reasons thus:—'Since I did not die, the Office lost nothing by me, and, as it has turned out, ran no risk: why, then, should they not restore me the premiums which I have paid?' To which it should be answered:—'Because the risk, which turned out favourably in your case, did not produce the same result in another case; and it is the very essence of an Insurance Office, that those who live pay for those who die. If you can induce the executors of those who have died during your tenure of your policy to refund what they have received from the Office, with compound interest, then the Office will repay your premiums, also with compound interest.'"

Similarly, when a policy is allowed to lapse, it is imagined by the public that the whole of the premiums paid becomes profit to the Office. But it is not so; only that portion becomes profit, by which the premium for the whole life exceeds the premium for the temporary insurance. Again let us hear what the learned Professor says:—

"Every premium which is paid by an insurer contains the consideration given for the chance of his dying in each and every subsequent year. If, then, he remain a member of the Office, and stand the risk of death during a certain number of years, all such part of his premiums as was consideration for the risks of those years became due to the Office, and was taken by the Office as compensation for those risks, and cannot, therefore, be said to fall to them as profit upon the lapse of the policy. Two individuals, A and B, go to the Office on the same day, and insure their lives for the same sum, A upon his whole life, and B for seven years; A pays say £10 of premium and B £7. At the end of seven years A allows his policy to lapse, just at the time when B's policy expires by its own construction. What does the Office gain by the lapse? Evidently the temporary annuity of £3, by which the two premiums differ. The £7 paid by A out of £10 is not more than sufficient to pay his share of the claims which arose during the years which he continued in the Office; the remaining £3 was a reserve for future years, which becomes profit to the Office on his declining to stand the risks of those years."

It does, no doubt, appear a hardship, that a man who, through illness or misfortune, is prevented from continuing his policy,

should, after years of saving, not only find that his family is after all left unprovided for, but that only a portion of the monies he has paid are returned to him. But is his case worse than that of a man who pays a watchman to guard his property? If, after a time, he finds it too expensive to keep this watchman, would he, on discharging him, claim a return of all the wages paid, because the property was not attacked by thieves and no actual risk arose? The wages of each week covered the expense of protecting the property during that week; and similarly, the greater portion of the premium paid goes to cover the assurance against the risk of death during the period for which it is paid.

Under the prevailing system of assurance, then, I do not see how any other course can be adopted, than that of allowing, in the case of the discontinuance of a policy, merely its surrender value, that is, the present value of the reversion, diminished by the value of an annuity of the premiums. Yet the utmost leniency and forbearance should be exercised by the Government Office, with a view to obviate the necessity of the working man abandoning his life policy, should he be in temporary embarrassment or suffer from illness. It would be well to imitate the example of the class Offices, who allow policies to be reinstated within a year after they have lapsed, rather than that of some Friendly and Industrial Societies, who, on the slightest irregularity in paying the premiums, at once cause the policies to become void.

Dr. Farr has propounded a plan similar to that, which we referred to when considering the subject of life annuities, namely, the establishment of a "Savings Insurance Bank." He says,*—"Young working men or servants can, as a general rule, only save a small sum out of their yearly earnings in the early part of life, and cannot therefore pay the single premium; much less can they engage to pay an annual premium for the whole period of life, part of which will probably be passed in sickness, infirmity, or straitened circumstances. To grant insurances on the plan of an annual premium, is also to incur much risk, as, if bad lives should happen to be admitted in undue proportion, with some such sum as £2. 3*s.* 7*d.* in hand on each life, the fund might have to pay, in the first year or two, several sums of £100."

The plan by which he proposes to meet the case, and to place assurance within the reach of those, who resort to Savings Banks, is:—

A single premium of £1,	at age	20,	insures	£2. 16*s.* 9*d.*	at death;
"	"	21,	"	£2. 15*s.* 11*d.*	"
"	"	22,	"	£2. 15*s.* 1*d.*	" &c.;

* Appendix to 12th Registrar-General's Report, p. xlv.

So that for the payment of £1 a year during 3 years, a man would stand insured for £8. 7*s.* 9*d.*; after 10 years, for £26. 10*s.* 1*d.*; and after 30 years, for £68. 4*s.* 5*d.*, having deposited £30 in premiums. This £1 a year might have been paid in monthly, or even smaller, instalments; and if in any year the assured should be unable to pay the premium, or wishes to discontinue paying altogether, the policy remains as before; only no additions continue to be made to the amount assured. According to this plan, so long as the premium in any year is not increased beyond the original amount, one investigation into the state of health of the depositor suffices.

Dr. Farr goes further, and contends that, under this plan, the premiums paid may be withdrawn in full by the depositors at any time, and thus Mr. Gladstone's desire of giving a liberal return to persons allowing their policies to lapse would be met.

This scheme is, no doubt, very ingenious, but I apprehend if, as Dr. Farr himself suggests, some addition is to be made to the tabular premiums for expenses of management and to cover fluctuations in interest, &c., a *full* return of the premiums paid cannot fairly be made to the assured in all cases. For instance, if, according to his plan, the premiums are loaded 20 per cent., then, for an annual payment of £1. 4*s.*, he stands assured, after 10 years time, for £26. 10*s.* The present value of this sum, for a life aged 30, is £10. 14*s.* 10*d.*, which is all that can be returned to the assured, whereas he has really paid the sum of £14. It is true, that the longer the assurance exists, the less the difference between the surrender value and the premiums paid will become. Yet I think this is rather an objection to a plan, which otherwise would have been particularly suited for a Government Assurance Office, occupying as it does an intermediate place between ordinary life assurance and the ordinary Savings Bank system.

I humbly beg to suggest a plan, which might meet the point upon which the Chancellor of the Exchequer lays so much stress, viz., to provide liberal terms to those who drop their policies at any time. The premiums for an assurance might, I think, be calculated to provide for a return of all premiums paid on the policy in the event of death. The formula for the annual premium is $\frac{M_x}{N_x - R_x}$, and the following is a table of the net annual and monthly rates for several ages, at 3 per cent. and according to the English Male Life Table, No. 2:—

Age.	Annual Premium.	Monthly Premium.
	£ s. d.	£ s. d.
16	2 12 5	0 4 7
20	3 0 4	0 5 3
25	3 12 2	0 6 4
30	4 7 10	0 7 8
35	5 9 1	0 9 6
40	6 18 5	0 12 1
45	9 0 3	0 15 9
50	12 2 1	1 1 2
55	16 18 0	1 9 7
60	24 10 6	2 2 11

It will, I believe, be found that, according to this scheme, for the more advanced ages, or where an assurance has been many years in force, the *total* amount of the premiums paid can be returned at any time to the assured—the usufruct of the interest realised upon the premiums paid, having been sufficient to cover the risk—and even in the most disadvantageous case, a large portion of the premium can with safety be returned.

It is true that the premiums, according to this method, will be found excessively high; but we must remember that, besides the advantages pointed out, not only will the amount assured be paid on the claim arising, but, according to the number of premiums paid, often three or four times the amount assured. Besides, Mr. Gladstone wishes us distinctly to understand, that if terms of particular liberality are to be offered to those who drop their policies, the premiums have to be adequately raised. I have calculated the above annual and monthly rates for several ages, according to this view.

If we give this interpretation to Mr. Gladstone's proposition respecting the lapsing of policies, no such absurdity as would appear *primâ facie* will be found to exist. It would not, however, be fair to allow Government, according to the scheme I propose, to grant policies for amounts at all near to £100, as, through the fact of the premiums being returnable, together with the amount assured at death, a much larger sum than the net amount contracted for is actually assured; to issue policies for such large amounts would be invading the province of the private Insurance Offices. Fifty pounds, I think, should be the maximum for this class of assurance.

I think that even, if the idea of making periodical returns of profits to the assured be not entertained, it would be desirable that

periodical valuations of the assurances granted by the Government be made. It may be, that such a valuation would show unexpected results, and it would, at all events, be the most trustworthy guide to satisfy the country, that the right path is being followed.

In the event of its being discovered that considerable profits are being realised, a reserve fund should be established, to serve as a guarantee for the nation that it will not be called upon at any future time—and only after this has accumulated for years, and the system is found to be working satisfactorily, should a division of profits be recommended.

I have before mentioned that, since the State guarantees the amounts assured, it acts the part of shareholders subscribing to a capital, who are indemnified for the guarantee they render by a portion of the profits being allotted to them. In the same way the State should receive a portion of the profits. Some proportion of the profits might, however, be returned to the assured, in the usual modes of increase to the sum assured, reduction of the premium, &c.

I may here mention, that the nation must be quite prepared to be called upon to subsidize the Assurance Office in the first few years of its existence, as the aggregate of the claims may exceed the total of the funds in hand. But this deficiency could easily be made up again in a succeeding year. This would happen, not through any extraordinary ill-fortune, but in the ordinary nature of assurance business.

I will not long detain you by dwelling upon the practical details of the scheme.

The management of the Government Assurance Office should, I think, be entrusted to a Board of Control, similar to that conducting the administration of the National Debt. It would, of course, be desirable to keep the accounts in every way perfectly distinct from those of the other Government departments.

The Head Office, whence all the instructions to the different branches would have to be issued, should be located in London.

By making use of the same machinery as that employed for the Post Office Savings Banks, between two and three thousand agencies, distributed equally throughout the country, would be at once created. It may be a question, whether the benefits of assurance should be equally extended to Scotland and Ireland as to England and Wales. I think, if proper care be exercised, there can be no objection thereto. Although the life table which I have suggested should be adopted, is based only on the mortality experienced in England, it will, for all purposes, apply to the whole of Great Britain and Ireland.

The medical examination should, as mentioned before, be entrusted to the Poor-Law medical officers, of whom there are nearly 4,000 distributed over the United Kingdom. It would also be desirable, that some medical authority of eminence should be located at the Head Office, whose duty it would be to examine the reports of the local medical examiners, and to exercise in other respects supervision over the medical department of the Government Life Assurance Office.

In the case of operatives, the employer or his foreman are the fittest persons to refer to respecting the health and habits of the life proposed. The latter, especially, is well able to testify to the eligibility for assurance of those employed under him.

The Chancellor of the Exchequer proposes not to receive instalments for smaller amounts than 2*s*. This, as I mentioned before, will hardly suit the working classes; but the difficulty might be met, if that large staff attached to the Post-Offices, namely the letter-carriers, were to collect the premiums weekly. The postmaster would only have to account once a month to the Head Office for the premiums received, and no complication would arise on that score.

In the larger towns it will, I think, be necessary to have an official whose sole duty it will be to look after the insurance business; and it will also be desirable that inspectors should occasionally be sent out from the Head Office to supervise the postmasters and medical officers.

In the event of a claim arising, it will be necessary that it should be paid as soon as possible after the decease of the life assured. The practice of Offices in delaying one or three months in paying claims under their policies is a source of but a very trifling profit indeed.

I think that, looking at the case of Savings Banks, where I believe there has not been a single case of fraud, there need be little fear of personation. The clergyman, the magistrate, the Poor-Law guardian, the tax-gatherer and registrar, would all aid the Government; and, most of all, the employer of labour could, by testifying to the identity of the claimant and the deceased, prevent any attempt at fraud being made. Where an assured has changed his place of abode, the measures to insure his identity should be even more stringent.

I have, in conclusion, to mention one more point.

The Chancellor of the Exchequer proposes, that no person entering into a contract for a payment on death, or any person

becoming beneficially interested therein, shall be exempted from probate or any stamp duty payable by law.

I have no doubt this is proposed with a view that the Government Assurance Office may not be more favoured than the other Assurance Companies. It has, however, been found, that for persons in humble circumstances the necessity of taking out probate or letters of administration is often attended with hardships and inconvenience. It would be giving a wholesome impetus to life assurance if, for policies under £100, this requirement were not to be insisted upon, the amount assured being paid to the next of kin, or the person that was nominated by the assured.

The rule on this subject adopted by the County of Kent Friendly Society—one of the best-conducted Friendly Societies in the kingdom, established so far back as 1828—is so to the point that I will quote it :—

"Burial money, upon the occasion of the death of the assured, shall, upon satisfactory proof of his or her death, be paid to his or her widow or husband; and if there be no widow or husband surviving, to his or her surviving child, if only one; or if children, to such children in equal parts; and if there be no child surviving, then to his or her father; and if there be no father surviving, then to his or her mother; and if there be no mother surviving, then to his or her surviving brother and sister, or brothers and sisters if more than one, in equal shares; and if only one, the whole to such one; and if there be no brother and sister, then to the person or persons who shall appear to the trustees to be entitled under the Statute of Distributions to receive the same.

"But burial money when due under provisions as above, shall, in whole or in parts as the case may be, be paid to any one or more of the persons specified as above, in preference to any other or others of such persons, provided that any such person or persons so preferred shall have been nominated by the assured to receive the same in a writing deposited by him or her with the Secretary of the Society previous to and remaining unrevoked at the time of his or her death. And every such nomination, to be valid and effective, shall be signed by the member making it; and his signature shall be attested by at least one witness, whose residence and calling shall be fully described. Any nomination as above may be at any time revoked; and upon every occasion of a nomination being revoked, the revocation to be effected shall be in writing, and signed and witnessed as in the case of a nomination being made, and shall be deposited with the Secretary previous to the death of the member. And in every case of a nomination being revoked, as well as in every case of the death of a nominee in the lifetime of the assurer, it shall be lawful to make a new nomination as before upon the payment of a fee for the same not exceeding 2*s.* 6*d.*"

Mr. Whitbread, in his Bill for a Poor's Insurance Fund, thought a like course desirable. In fairness a similar privilege would have to

be conceded to those assuring for amounts under £100 both in the Class and Industrial Assurance Offices.

I have now fully entered into the details of the Government Bill, and answered, I trust satisfactorily, the objections that have been raised against it. I fear that much that I have said may be considered trite; but though I was well aware that many of my remarks might not be new to actuaries, yet, thinking that this paper might come into the hands of those, who are less informed upon the subject, I thought it desirable not to omit what might contribute to the proper understanding of life assurance as proposed to be undertaken by Government. Some of the particulars of the scheme may be open to improvement and revision; but in its general features I believe, if adopted, it will be productive of considerable benefit to the large mass of the working population.

A Budget of Paradoxes. *By* Professor De Morgan.

(Continued from page 284, vol. xi.)

An essay to ascertain the value of leases and annuities for years and lives. By W[eyman] L[ee]. London, 1737, 8vo.

A valuation of Annuities and Leases certain, for a single life. By Weyman Lee, Esq., of the Inner Temple. London, 1751, 8vo. Third edition, 1773.

Every branch of exact science has its paradoxer. The world at large cannot tell with certainty who is right in such questions as squaring the circle, &c. Mr. Weyman Lee was the assailant of what all who had studied called demonstration in the question of annuities; and he can be exposed to the world: for his error arose out of his not being able to see that the whole is the sum of all its parts.

By an annuity, say of £100, now bought, is meant that the buyer is to have for his money £100 in a year if he be then alive; £100 at the end of two years, if then alive; and so on. It is clear that he would buy a life annuity if he should buy the first £100 in one Office, the second in another; and so on. All the difference between buying the whole from one Office, and buying all the separate contingent payments at different Offices, is immaterial to calculation. Mr. Lee would have agreed with the rest of the world about the payments to be made to the several different Offices, in consideration of their several contracts: but he differed from every one else about the sum to be paid to *one* Office. He contended that the way to value an annuity is to find out the term

of years which the individual has an even chance of surviving, and to charge for the life annuity the value of an annuity certain for that term.

It is very common to say that Lee took the average life, or expectation, as it is wrongly called, for his term: and this I have done myself, taking the common story. Having exposed the absurdity of this second supposition, taking it for Lee's, in my *Formal Logic*, I will now do the same with the first.

A mathematical truth is true in its extreme cases. Lee's principle is that an annuity on a life is the annuity made certain for the term within which it is an even chance the life drops. If, then, of a thousand persons, 500 be sure to die within a year, and the other 500 be immortal, Lee's price of an annuity to any one of these persons is the present value of one payment: for one year is the term which each one has an even chance of surviving and not surviving. But the true value is obviously half that of a perpetual annuity: so that at 5 per cent., Lee's rule would give less than the tenth of the true value. It must be said for the poor circle-squarers, that they never err so much as this.

Lee would have said, if alive, that I have put an *extreme case*: but any *universal* truth is true in its extreme cases. It is not fair to bring forward an extreme case against a person who is speaking as of usual occurrences: but it is quite fair when, as frequently happens, the proposer insists upon a perfectly general acceptance of his assertion. And yet many who go the whole hog protest against being tickled with the tail. Counsel in court are good instances: they are paradoxers by trade. June 13, 1849, at Hertford, there was an action about a ship insured against a *total* loss: some planks were saved, and the underwriters refused to pay. Mr. Z. (for deft.), "There can be no degrees of totality; and some timbers were saved."—L.C.B. "Then if the vessel were burned to the water's edge, and some rope saved in the boat, there would be no total loss."—Mr. Z. "This is putting a very extreme case."—L.C.B. "The argument would go that length." What would *Judge* Z.—as he now is—say to the extreme case beginning somewhere between six planks and a bit of rope?

No. VIII. 1754—1792.

Histoire des recherches sur la quadrature du cercle....avec une addition concernant les problèmes de la duplication du cube et de la trisection de l'angle. Paris, 1754, 12mo. [By Montucla.]

This is *the* history of the subject. It was a little episode to the great history of mathematics by Montucla, of which the first

edition appeared in 1758. There was much addition at the end of the fourth volume of the second edition; this is clearly by Montucla, though the bulk of the volume is put together, with help from Montucla's papers, by Lalande. There is also a second edition of the history of the quadrature, Paris, 1831, 8vo., edited, I think, by Lacroix; of which it is the great fault that it makes hardly any use of the additional matter just mentioned.

Montucla is an admirable historian when he is writing from his own direct knowledge: it is a sad pity that he did not tell us when he was depending on others. We are not to trust a quarter of his book, and we must read many other books to know which quarter. The fault is common enough, but Montucla's good three-quarters is so good that the fault is greater in him than in most others: I mean the fault of not acknowledging; for an historian cannot read everything. But it must be said that mankind give little encouragement to candour on this point. Hallam, in his *History of Literature,* states with his own usual instinct of honesty every case in which he depends upon others: Montucla does not. And what is the consequence? Montucla is trusted, and believed in, and cried up in the bulk; while the smallest talker can lament that Hallam should be so unequal and apt to depend on others, without remembering to mention that Hallam himself gives the information. As to a universal history of any great subject being written entirely upon primary knowledge, it is a thing of which the possibility is not yet proved by an example. Delambre attempted it with astronomy, and was removed by death before it was finished to say nothing of the gaps he left.

Montucla was nothing of a bibliographer, and his descriptions of books in the first edition were insufficient. The Abbé Rive fell foul of him, and, as the phrase is, gave it him. Montucla took it with great good humour, tried to mend, and, in his second edition, wished his critic had lived to see the *vernis de bibliographe* which he had given himself.

I have seen Montucla set down as an *esprit fort,* more than once: wrongly, I think. When he mentions Barrow's address to the Almighty, he adds, "On voit, au reste, par là, que Barrow étoit un pauvre philosophe; car il croyait en l'immortalité de l'âme, et en une Divinité autre que la nature universelle." This is a sneer, not an expression of opinion. In the book of mathematical recreations which Montucla constructed upon that of Ozanam, and Ozanam upon that of Van Etten, now best known in England by Hutton's similar treatment of Montucla, there is an amusing chapter on the

quadrators. Montucla refers to his own anonymous book of 1754 as a curious book published by Jombert.

Antinewtonianismus. By Cælestino Cominale, M.D. Naples, 1754 and 1756, 2 vols. 4to.

The first volume upsets the theory of light; the second vacuum, vis inertiæ, gravitation, and attraction. I confess I never attempted these big Latin volumes, numbering 450 closely-printed quarto pages. The man who slays Newton in a pamphlet is the man for me. But I will lend them to anybody who will give security, himself in £500, and two sureties in £250 each, that he will read them through, and give a full abstract; and I will not exact security for their return. I have never seen any mention of this book: it has a printer, but not a publisher, as happens with so many unrecorded books.

1755. The French Academy of Sciences came to the determination not to examine any more quadratures or kindred problems. This was the consequence, no doubt, of the publication of Montucla's book: the time was well chosen; for that book was a full justification of the resolution. The Royal Society followed the same course, I believe, a few years afterwards. When our Board of Longitude was in existence, most of its time was consumed in listening to schemes, many of which included the quadrature of the circle. It is certain that many quadrators have imagined the longitude problem to be connected with theirs: and no doubt the notion of a reward being offered by Government for a true quadrature is a result of the reward offered for the longitude. Let it also be noted that this longitude reward was not a premium upon excogitation of a mysterious difficulty. The legislature was made to know that the rational hopes of the problem were centered in the improvement of the lunar tables and the improvement of chronometers. To these objects alone, and by name, the offer was directed; several persons gained rewards for both; and the offer was finally repealed.

Fundamentalis Figura Geometrica, primas tantum lineas circuli quadraturæ possibilitatis ostendens. By Niels Erichsen (Nicolaus Ericius), shipbuilder, of Copenhagen. Copenhagen, 1755, 12mo.

The quadrature is not worth notice. Erichsen is the only squarer I have met with who has distinctly asserted the particulars of that reward which has been so frequently thought to have been offered in England. He says that in 1747, the Royal Society, on the second of June, offered to give a large reward for the quadra-

ture of the circle and a true explanation of magnetism, in addition to £30,000 previously promised for the same. I need hardly say that the Royal Society had not £30,000 at that time, and would not, if it had had such a sum, have spent it on the circle, nor on magnetic theory; nor would it have coupled the two things. On this book see *Notes and Queries*, 1st S., xii. 306. Perhaps Erichsen meant that the £30,000 had been promised by the Government, and the addition by the Royal Society.

Theoria Philosophiæ Naturalis redacta ad unicam legem virium in natura existentium. Editio *Veneta* prima. By Roger Joseph Boscovich. Venice, 1763, 4to.

The first edition is said to be of Vienna, 1758. This is a celebrated work on the molecular theory of matter, grounded on the hypothesis of spheres of alternate attraction and repulsion. Boscovich was a Jesuit of varied pursuit. During his measurement of a degree of the meridian, while on horseback or waiting for his observations, he composed a Latin poem of about five thousand verses on eclipses, with notes, which he dedicated to the Royal Society: *De Solis et Lunæ defectibus*; London, Millar and Dodsley, 1760, 4to.

Traité de paix entre Des Cartes et Newton, *précédé* des vies littéraires de ces deux chefs de la physique moderne.... By Aimé Henri Paulian. Avignon, 1763, 12mo.

I have had these books for many a year without feeling the least desire to see how a lettered Jesuit would atone Descartes and Newton. On looking at my two volumes, I find that one contains nothing but the literary life of Des Cartes; the other nothing but the literary life of Newton. The preface indicates more: and Watt mentions *three* volumes. I dare say the first two contain all that is valuable. On looking more attentively at the two volumes, I find them both readable and instructive; the account of Newton is far above that of Voltaire, but not so popular. But he should not have said that Newton's family came from Newton in Ireland. Sir Rowland Hill gives fourteen *Newtons* in Ireland: twice the number of the cities that contended for the birth of Homer may now contend for the origin of Newton, on the word of Father Paulian.

Philosophical Essays, in three parts. By R. Lovett, Lay Clerk of the Cathedral Church of Worcester. Worcester, 1766, 8vo.

The electrical philosopher: containing a new system of physics founded upon the principle of an universal Plenum of elementary fire.... By R. Lovett. Worcester, 1774, 8vo.

Mr. Lovett was one of those ether philosophers who bring in elastic fluid as an explanation by imposition of words, without

deducing any one phenomenon from what we know of it. And yet he says that attraction has received no support from geometry; though geometry, applied to a particular law of attraction, had shown how to predict the motions of the bodies of the solar system. He, and many of his stamp, have not the least idea of the confirmation of a theory by accordance of deduced results with observation posterior to the theory.

Lettres sur l'Atlantide de Platon, et sur l'ancien Histoire de l'Asie, pour servir de suite aux lettres sur l'origine des Sciences, addressées à M. de Voltaire, par M. Bailly. London and Paris, 1779, 8vo.

I might enter here all Bailly's histories of astronomy. The paradox which runs through them all, more or less, is the doctrine that astronomy is of immense antiquity, coming from some forgotten source, probably the drowned island of Plato, peopled by a race whom Bailly makes, as has been said, to teach us everything except their existence and their name. These books, the first scientific histories which belong to readable literature, made a great impression by power of style: Delambre created a strong reaction, of injurious amount, in favour of history founded on contemporary documents, which early astronomy cannot furnish. These letters are addressed to Voltaire, and continue the discussion. There is one letter of Voltaire, being the fourth, dated Feb. 27, 1777, and signed "le vieux malade de Ferney, V. puer centum annorum." Then begin Bailly's Letters, from January 16 to May 12, 1778. From some ambiguous expressions in the Preface, it would seem that these are fictitious Letters, supposed to be addressed to Voltaire at their dates. Voltaire went to Paris Feb. 10, 1778, and died there May 30. Nearly all this interval was his closing scene, and it is very unlikely that Bailly would have troubled him with these Letters.

An inquiry into the cause of motion, or a general theory of physics. By S. Miller. London, 1781, 4to.

Newton all wrong: matter consists of two kinds of particles, one inert, the other elastic and capable of expanding themselves *ad infinitum*.

Method to discover the difference of the earth's diameters; proving its true ratio to be not less variable than as 45 is to 46, and shortest in in its pole's axis 174 miles....likewise a method for fixing an universal standard for weights and measures. By Thomas Williams. London, 1788, 8vo.

Mr. Williams was a paradoxer in his day, and proposed what was, no doubt, laughed at by some. He proposed the plan which

the French—independently of course—carried into effect a few years after. He would have the 52nd degree of latitude divided into 100,000 parts, and each part a geographical yard. The geographical tun was to be the cube of the geographical yard filled with sea-water taken some leagues from land. All multiples and subdivisions were to be decimal.

I was beginning to look up those who had made similar proposals, when a learned article on the proposal of a metrical system came under my eye in the *Times* of Sept. 15, 1863. The author cites Mouton, who would have the minute of a degree divided into 10,000 *virgulæ;* James Cassini, whose foot was to be six thousandths of a minute; and Paucton, whose foot was the 400,000th of a degree. I have verified the first and third statements: surely the second ought to be the *six-thousandth.*

An inquiry into the Copernican system....wherein it is proved, in the clearest manner, that the earth has only her diurnal motion....with an attempt to point out the only true way whereby mankind can receive any real benefit from the study of the heavenly bodies. By John Cunningham. London, 1789, 8vo.

The "true way" appears to be the treatment of heaven and earth as emblematical of the Trinity.

Cosmology. An inquiry into the cause of what is called gravitation or attraction, in which the motions of the heavenly bodies, and the preservation and operations of all nature, are deduced from an universal principle of efflux and reflux. By T. Vivian, vicar of Cornwood, Devon. Bath, 1792, 12mo.

Attraction, an influx of matter to the sun; centrifugal force, the solar rays; cohesion, the pressure of the atmosphere. The confusion about centrifugal *force,* so called, as demanding an external agent, is very common.

No. IX. 1792—1802.

The Commentaries of Proclus. Translated by Thomas Taylor. London, 1792, 2 vols. 4to.

The reputation of "the Platonist" begins to grow, and will continue to grow. The most authentic account is in the *Penny Cyclopædia,* written by one of the few persons who knew him well, and one of the fewer who possess all his works. At page lvi. of the Introduction is Taylor's notion of the way to find the circumference. It is not geometrical, for it proceeds on the motion of a point: the words, "on account of the simplicity of the impulsive motion, such a line must be either straight or circular," will suffice

to show how Platonic it is. Taylor certainly professed a kind of heathenism. D'Israeli said, "Mr. T. Taylor, the Platonic philosopher and the modern Plethon, consonant to that philosophy, professes polytheism." Taylor printed this in large type, in a page by itself after the dedication, without any disavowal. I have seen the following, Greek and translation both, in his handwriting:—"Πας ἀγαθος ᾗ ἀγαθος ἐθνικος· και πας χριστιανος ᾗ χριστιανος κακος. Every good man, so far as he is a good man, is a heathen; and every Christian, so far as he is a Christian, is a bad man." Whether Taylor had in his head the Christian of the New Testament, or whether he drew from those members of the "religious world" who make manifest the religious flesh and the religious devil, cannot be decided by us, and perhaps was not known to himself. If a heathen, he was a virtuous one.

The principles of Algebra. By William Frend. London, 1796, 8vo. Second Part, 1799.

This algebra, says Dr. Peacock, shows "great distrust of the results of algebraical science which were in existence at the time when it was written." Truly it does: for, as Dr. Peacock had shown by full citation, it makes war of extermination upon all that distinguishes algebra from arithmetic. Robert Simson and Baron Maseres were Mr. Frend's predecessors in this opinion.

The genuine respect which I entertained for my father-in-law did not prevent my canvassing with perfect freedom his anti-algebraical and anti-Newtonian opinions, in a long obituary memoir read at the Astronomical Society in February, 1842, which was written by me. It was copied into the *Athenæum* of March 19. It must be said that if the manner in which algebra *was* presented to the learner had been true algebra, he would have been right: and if he had confined himself to protesting against the imposition of attraction as a fundamental part of the existence of matter, he would have been in unity with a great many, including Newton himself. I wish he had preferred amendment to rejection, when he was a college tutor: he wrote and spoke English with a clearness which is seldom equalled.

His anti-Newtonian discussions are confined to the preliminary chapters of his *Evening Amusements*, a series of astronomical lessons in 19 volumes, following the moon through a period of the golden numbers.

There is a mistake about him which can never be destroyed. It is constantly said that, at his celebrated trial in 1792, for sedition and opposition to the liturgy, &c., he was *expelled* the University.

He was *banished.* People cannot see the difference; but it made all the difference to Mr. Frend. He held his fellowship and its profits till his marriage in 1808, and was a member of the University and of its Senate till his death in 1841, as any Cambridge Calendar up to 1841 will show. That they would have expelled him if they could, is perfectly true; and there is a funny story—also perfectly true—about their first proceedings being under a statute which would have given the power, had it not been discovered during the proceedings that the statute did not exist. It had come so near to existence as to be entered into the Vice-Chancellor's book for his signature, which it wanted, as was not seen till Mr. Frend exposed it: in fact, the statute had never actually passed.

There is an absurd mistake in Gunning's *Reminiscences of Cambridge.* In quoting a passage of Mr. Frend's pamphlet, which was very obnoxious to the existing government, it is printed that the poor market-women complained that they were to be *scotched* a quarter of their wages by taxation; and attention is called to the word by its being three times printed in italics. In the pamphlet it is "sconced"; that very common old word for fined or mulcted.

Lord Lyndhurst, who has just passed away under a load of years and honours, was Mr. Frend's private pupil at Cambridge. At the time of the celebrated trial, he and two others amused themselves, and vented the feeling which was very strong among the undergraduates, by chalking the walls of Cambridge with "Frend for ever." While thus engaged in what, using the term legally, we are probably to call his first publication, he and his friends were surprised by the proctors. Flight and chase followed of course: Copley and one of the others escaped; the third, whose name I forget, but who afterwards, I have been told, was a bishop, was captured and impositioned. Looking at the Cambridge Calendar to verify the fact that Copley was an undergraduate at the time, I find that there are but two other men in the list of honours in his year, whose names are now widely remembered. And they were both celebrated schoolmasters; Butler of Harrow, and Tate of Richmond.

A treatise on the sublime science of heliography, satisfactorily demonstrating our great orb of light, the sun, to be absolutely no other than a body of ice! Overturning all the received systems of the universe hitherto extant; proving the celebrated and indefatigable Sir Isaac Newton, in his theory of the solar system, to be as far distant from the truth, as any of the heathen authors of Greece or Rome. By Charles Palmer, Gent. London, 1798, 8vo.

Mr. Palmer burned some tobacco with a burning glass, saw that a lens of ice would do as well, and then says—

"If we admit that the sun could be removed, and a terrestrial body of ice placed in its stead, it would produce the same effect. The sun is a crystaline body receiving the radience of God, and operates on this earth in a similar manner as the light of the sun does when applied to a convex mirror or glass."

The mathematical and philosophical works of the Right Rev. John Wilkins....In two volumes. London, 1802, 8vo.

This work, or at least part of the edition—all for aught I know—is printed on wood; that is, on paper made from wood-pulp. It has a rough surface, and when held before a candle is of very unequal transparency. There is in it a reprint of the works on the earth and moon. The discourse on the possibility of going to the moon, in this and the edition of 1640, is incorporated: but from the account in the life prefixed, and a mention by D'Israeli, I should suppose that it had originally a separate title-page, and some circulation as a separate tract. Wilkins treats this subject half seriously, half jocosely; he has evidently not quite made up his mind. He is clear that "arts are not yet come to their solstice," and that posterity will bring hidden things to light. As to the difficulty of carrying food, he thinks, scoffing Puritan that he is, the Papists may be trained to fast the voyage, or may find the bread of their Eucharist "serve well enough for their *viaticum*." He also puts the case that the story of Domingo Gonzales may be realized, namely, that wild geese find their way to the moon. It will be remembered—to use the usual substitute for, It has been forgotten—that the posthumous work of Bishop Francis Godwin of Llandaff was published in 1638, the very year of Wilkins's first edition, in time for him to mention it at the end. Godwin makes Domingo Gonzales get to the moon in a chariot drawn by wild geese, and, as old books would say, discourses fully on that head. It is not a little amusing that Wilkins should have been seriously accused of plagiarizing Godwin, Wilkins writing in earnest, or nearly so, and Godwin writing fiction. It may serve to show philosophers how very near pure speculation comes to fable. From the sublime to the ridiculous there is but a step: which is the sublime, and which the ridiculous, every one must settle for himself. With me, good fiction is the sublime, and bad speculation the ridiculous. The number of bishops in my list is small. I might, had I possessed the book, have opened the list of quadrators with an Archbishop of Canterbury, or at least with a divine who was not wholly not archbishop. Thomas Bradwardine (Bragvardinus, Bragadinus) was elected in 1348; the Pope put in another, who

died unconsecrated; and Bradwardine was again elected in 1349, and lived five weeks longer, dying, I suppose, unconfirmed and unconsecrated. He squared the circle, and his performance was printed at Paris in 1494. I have never seen it, nor any work of the author, except a tract on proportion.

No. X. 1803—1819.

Nouvelle théorie des parallèles. Par Adolphe Kircher [so signed at the end of the appendix]. Paris, 1803, 8vo.

An alleged emendation of Legendre. The author refers to attempts by Hoffman, 1801, by Hauff, 1799, and to a work of Karsten, or at least a theory of Karsten, contained in "*Tentamen novæ parallelarum theoriæ notione situs fundatæ;* auctore G. C. Schwal, Stuttgardæ, 1801, en 8 volumes." Surely this is a misprint; *eight* volumes on the theory of parallels? If there be such a work, I trust I and it may never meet, though ever so far produced.

Soluzione.... della quadratura del Circolo. By Gaetano Rossi. London, 1804, 8vo.

The three remarkable points of this book are, that the household of the Prince of Wales took ten copies, Signora Grassini sixteen, and that the circumference is $3\frac{1}{5}$ diameters. That is, the appetite of Grassini for quadrature exceeded that of the whole household (*loggia*) of the Prince of Wales in the ratio in which the semicircumference exceeds the diameter. And these are the first two in the list of subscribers. Did the author see this theorem?

An appeal to the republic of letters in behalf of injured science, from the opinions and proceedings of some modern authors of elements of geometry. By George Douglas. Edinburgh, 1810, 8vo.

Mr. Douglas was the author of a very good set of mathematical tables, and of other works. He criticizes Simson, Playfair, and others—sometimes, I think, very justly. There is a curious phrase, which occurs more than once. When he wants to say that something or other was done before Simson or another was born, he says, "before he existed, at least as an author." He seems to reserve the possibility of Simson's *pre-existence*, but at the same time to assume that he never wrote anything in his previous state. Tell me that Simson pre-existed in any other way than as editor of some pre-existent Euclid? Tell Apella!

Philosophia Sacra, or the principles of natural Philosophy. Extracted from Divine Revelation. By the Rev. Samuel Pike. Edited by the Rev. Samuel Kittle. Edinburgh, 1815, 8vo.

This is a work of modified Hutchinsonianism, which I have seen cited by several. Though rather dark on the subject, it seems not to contradict the motion of the earth, or the doctrine of gravitation. Mr. Kittle gives a list of some Hutchinsonians—as Bishop Horne; Dr. Stukeley; the Rev. W. Jones, author of *Physiological Disquisitions;* Mr. Spearman, author of *Letters on the Septuagint,* and editor of Hutchinson; Mr. Barker, author of *Reflexions on Learning;* Dr. Catcott, author of a work on the creation, &c.; Dr. Robertson, author of a *Treatise on the Hebrew Language;* Dr. Holloway, author of *Originals, Physical and Theological;* Dr. Walter Hodges, author of a work on *Elohim;* Lord President Forbes (ob. 1747).

Sir Richard Phillips (born 1768) was conspicuous in 1793, when he was sentenced to a year's imprisonment for selling Paine's *Rights of Man;* and again when, in 1807, he was knighted as Sheriff of London. As a bookseller, he was able to enforce his astronomical opinions in more ways than others. For instance, in James Mitchell's *Dictionary of the Mathematical and Physical Sciences,* 1823, 12mo., which, though he was not technically a publisher, was printed for him—a book I should recommend to the collector of works of reference—there is a temperate description of his doctrines, which one may almost swear was one of his conditions previous to undertaking the work. Phillips himself was not only an anti-Newtonian, but carried to a fearful excess the notion that statesmen and Newtonians were in league to deceive the world. He saw this plot in Mrs. Airy's pension, and in Mrs. Somerville's. In 1836, he did me the honour to attempt my conversion. In his first letter he says:—

> "Sir Richard Phillips has an inveterate abhorrence of all the pretended wisdom of philosophy derived from the monks and doctors of the middle ages, and not less of those of higher name who merely sought to make the monkish philosophy more plausible, or so to disguise it as to mystify the mob of small thinkers."

So little did his writings show any knowledge of antiquity, that I strongly suspect, if required to name one of the monkish doctors, he would have answered—Aristotle. These schoolmen, and the "philosophical trinity of gravitating force, projectile force, and void space," were the bogies of his life.

I think he began to publish speculations in the *Monthly Magazine* (of which he was editor) in July, 1817: these he republished

separately in 1818. In the Preface, perhaps judging the feelings of others by his own, he says that he "fully expects to be vilified, reviled, and anathematized, for many years to come." Poor man! he was let alone. He appeals with confidence to the "impartial decision of posterity"; but posterity does not appoint a hearing for one per cent. of the appeals which are made; and it is much to be feared that an article in such a work of reference as this will furnish nearly all her materials fifty years hence. The following, addressed to M. Arago, in 1835, will give posterity as good a notion as she will probably need:—

"Even the present year has afforded EVER-MEMORABLE examples, paralleled only by that of the Romish Conclave who persecuted Galileo. Policy has adopted that maxim of Machiavel which teaches that it is *more prudent to reward* partisans than to *persecute* opponents. Hence, a bigotted party had influence enough with the late short-lived administration [I think he is wrong as to the administration] of Wellington, Peel, &c., to confer munificent royal pensions on three writers whose sole distinction was their advocacy of the Newtonian philosophy. A Cambridge professor last year published an elaborate volume in illustration of *Gravitation*, and on him has been conferred a pension of £300 per annum. A lady has written a light popular view of the Newtonian Dogmas, and she has been complimented by a pension of £200 per annum. And another writer, who has recently published a volume to prove that the only true philosophy is that of Moses, has been endowed with a pension of £200 per annum. Neither of them were needy persons, and the political and ecclesiastical bearing of the whole was indicated by another pension of £300 bestowed on a political writer, the advocate of all abuses and prejudices. Whether the conduct of the Romish Conclave was more base for visiting with legal penalties the promulgation of the doctrines that the Earth turns on its axis and revolves around the Sun; or that of the British Court, for its craft in conferring pensions on the opponents of the plain corollary, that all the motions on the earth are 'part and parcel' of these great motions, and those again and all like them consecutive displays of still greater motions in equality of action and reaction, is A QUESTION which must be reserved for the casuists of other generations.... I cannot expect that on a sudden you and your friends will come to my conclusion, that the present philosophy of the Schools and Universities of Europe, based on faith in witchcraft, magic, &c., is a system of execrable nonsense, *by which quacks live on the faith of fools;* but I desire a free and fair examination of my Aphorisms, and if a few are admitted to be true, merely as courteous concessions to arithmetic, my purpose will be effected, for men will thus be led to think; and if they think, then the fabric of false assumptions, and degrading superstitions will soon tumble in ruins."

This for posterity. For the present time I ground the fame of Sir R. Phillips on his having squared the circle without knowing it, or intending to do it. In the *Protest* presently noted, he discovered that "the force taken as 1 is equal to the sum of all its fractions...... thus $1 = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$, &c., carried to infinity."

This the mathematician instantly sees is equivalent to the theorem that the circumference of any circle is double of the diagonal of the cube on its diameter.

I have examined the following works of Sir R. Phillips, and heard of many others.

Essays on the proximate mechanical causes of the general phenomena of the Universe. 1818, 12mo.

Protest against the prevailing principles of natural philosophy, with the development of a common sense system (no date, 8vo. pp. 16).

Four dialogues between an Oxford Tutor and a disciple of the common-sense philosophy, relative to the proximate causes of material phenomena. 8vo. 1824.

A century of original aphorisms on the proximate causes of the phenomena of nature. 1835, 12mo.

Sir Richard Phillips had four valuable qualities; honesty, zeal, ability, and courage. He applied them all to teaching matters about which he knew nothing; and gained himself an uncomfortable life and a ridiculous memory.

Astronomy made plain: or only way the true perpendicular distance of the Sun, Moon, or Stars, from the earth, can be obtained. By Wm. Wood. Chatham, 1819, 12mo.

"If this theory be true, it will follow, of course, that this earth is the only one God made, and that it does not whirl round the sun, but *vice versâ* the sun round it."

No. XI. 1819—1825.

Historic doubts relative to Napoleon Buonaparte. London, 1819, 8vo.

This tract has since been acknowledged by Archbishop Whately and reprinted. It is certainly a paradox: but differs from most of those in my list as being a joke, and a satire upon the reasoning of those who cannot receive narrative, no matter what the evidence, which is to them utterly improbable *à priori*. But had it been serious earnest, it would not have been so absurd as many of those which I have brought forward. The next on the list is not a joke.

The idea of the satire is not new. Dr. King, in the dispute on the genuineness of Phalaris, proved with humour that Bentley did not write his own dissertation. An attempt has lately been made, for the honour of Moses, to prove, without humour, that Bishop Colenso did not write his own book. This is intolerable: anybody who tries to use such a weapon without banter, plenty and good, and of form suited to the subject, should get the drubbing which the poor man got in the Oriental tale for striking the dervishes with the wrong hand.

The excellent and distinguished author of this tract has ceased

to live. I call him the Paley of our day: with more learning and more purpose than his predecessor; but perhaps they might have changed places if they had changed centuries. The clever satire above named is not the only work which he published without his name. The following was attributed to him, I believe rightly: *Considerations on the Law of Libel, as relating to Publications on the subject of Religion*, by John Search. London, 1833, 8vo. This tract excited little attention: for those who should have answered it could not. Moreover, it wanted a prosecution to call attention to it: the fear of calling such attention may have prevented prosecutions. Those who have read it will have seen why.

Voltaire Chrétien; preuves tirées de ses ouvrages. Paris, 1820, 12mo.

If Voltaire has not succeeded in proving himself a strong theist, and a strong anti-revelationist, who is to succeed in proving himself one thing or the other in any matter whatsoever? By occasional confusion between theism and Christianity; by taking advantage of the formal phrases of adhesion to the Roman Church, which very often occur, and are often the happiest bits of irony in an ironical production; by citations of his morality, which is decidedly Christian, though often attributed to Brahmins; and so on—the author makes a fair case for his paradox, in the eyes of those who know no more than he tells them. If he had said that Voltaire was a better Christian than himself knew of, towards all mankind except men of letters, I for one should have agreed with him.

Address of M. Hoene Wronski to the British Board of Longitude, upon the actual state of the mathematics, their reform, and upon the new celestial mechanics, giving the definitive solution of the problem of longitude. London, 1820, 8vo.

M. Wronski was the author of seven quartos on mathematics, showing very great power of generalization. He was also deep in the transcendental philosophy, and had the Absolute at his fingers' ends. All this knowledge was rendered useless by a persuasion that he had greatly advanced beyond the whole world, with many hints that the Absolute would not be forthcoming, unless prepaid. He was a man of the widest extremes. At one time he desired people to see all possible mathematics in

$$Fx = A_0\Omega_0 + A_1\Omega_1 + A_2\Omega_2 + A_3\Omega_3 + \&c.,$$

which he did not explain, though there is meaning to it in the quartos. At another time he was proposing the general solution of the fifth degree by help of 625 independent equations of one

form and 125 of another. The first separate memoir from any Transactions that I ever possessed was given to me when at Cambridge; the refutation (1819) of this asserted solution, presented to the Academy of Lisbon by Evangelista Torriano. I cannot say I read it. The tract above is an attack on modern mathematicians in general, and on the Board of Longitude and Dr. Young.

De Attentionis mensura causisque primariis. By J. F. Herbart. Kœnigsberg, 1822, 4to.

This celebrated philosopher maintained that mathematics ought to be applied to psychology, in a separate tract, published also in 1822: the one above seems, therefore, to be his challenge on the subject. It is on *attention*, and I think it will hardly support Herbart's thesis. As a specimen of his formulæ, let t be the time elapsed since the consideration began, β the whole perceptive intensity of the individual, ϕ the whole of his mental force, and z the force given to a notion by attention during the time t. Then,

$$z=\phi(1-\varepsilon^{-\beta t}).$$

Now for a test. There is a *jactura*, v, the meaning of which I do not comprehend. If there be anything in it, my mathematical readers ought to interpret it from the formula

$$v=\frac{\pi\phi\beta}{1-\beta}\varepsilon^{-\beta t}+C\varepsilon^{-t};$$

and to this task I leave them, wishing them better luck than mine. The time may come when other manifestions of mind, besides *belief*, shall be submitted to calculation: at that time, should it arrive, a final decision may be passed upon Herbart.

The theory of the Whizgig considered; in as much as it mechanically exemplifies the three working properties of nature; which are now set forth under the guise of this toy, for children of all ages. London, 1822, 12mo. (pp. 24, B. McMillan, Bow Street, Covent Garden.)

The toy called the *whizgig* will be remembered by many. The writer is a follower of Jacob Behmen, William Law, Richard Clarke, and Eugenius Philalethes. Jacob Behmen first announced the three working properties of nature, which Newton stole, as described in the *Gentleman's Magazine*, July, 1782, p. 329. These laws are illustrated in the whizgig. There is the harsh astringent, attractive compression; the bitter compunction, repulsive expansion; and the stinging anguish, duplex motion. The author hints that he has written other works, to which he gives no clue. I have heard that Behmen was pillaged by Newton, and Swedenborg by Laplace, and Pythagoras by Copernicus, and Epicurus by Dalton, &c. I do not

think this mention will revive Behmen; but it may the whizgig, a very pretty toy, and philosophical withal, for few of those who used it could explain it.

A Grammar of infinite forms; or the mathematical elements of ancient philosophy and mythology. By Wm. Howison. Edinburgh, 1823, 8vo.

A curious combination of geometry and mythology. Perseus, for instance, is treated under the head "the evolution of diminishing hyperbolic branches."

(*To be continued.*)

CORRESPONDENCE.

ON A NOTATION TO BE USED IN LIFE ASSURANCE COMPUTATIONS.

To the Editor of the Assurance Magazine.

Sir,—Perhaps it is not void of interest to the readers of the *Magazine*, to take notice of a proposal of notation in life assurance computations, which has been agreed upon last autumn by some German actuaries. I believe it is just as much felt with you as it is with us, how disagreeable it is that different symbols are used by different authors for the same expression, and different meanings are given to the same symbol by different authors. It would be very desirable to have uniform symbols for life assurance computations, and we have tried to obtain it, at least for the most common and constantly used expressions. We perfectly know that the proposed way of notation is not always so simple as we should wish it to be, but it having been our principal object to make it agree with mathematical notation, we have laid principal stress upon the functional notation, thus facilitating the use of analytical methods of computation. As far as possible we have subordinated our selection of the symbols to the notation already in use principally in Germany, but we considered it of absolute necessity to reject as a symbol for life assurance computation any letter which has a generally acknowledged meaning in mathematics, for even those mathematical forms which until now do not enter in life assurance investigations may very likely be employed as soon as our computations are becoming of a more analytical form. Thus, π as the proportion of the diameter to the circumference of the circle, e as the base of the natural logarithmic system, d as the symbol of differentiation, l as the abbreviation of log. nat., i equal to $\sqrt{-1}$, were positively excluded. I annex a translation of the proposal, and some remarks added to it, and am,

Sir,

Yours most obediently,

Hamburgh, July, 1864. WILHELM LAZARUS.

Proposal for the adoption of uniform symbols in life assurance computations.

It is willingly acknowledged that the want of uniformity in the symbols used in life assurance computations is exceedingly disagreeable. Every

new work which touches this branch requires a new study of its notation; and this study becomes the more difficult, as a small collection of symbols is employed with constantly changing significations.

It is very desirable that the most common expressions obtain a fixed notation, used by all authors, and the following proposal is therefore recommended for general adoption.

We have made it a leading point in our proposal that the notation being of mathematical nature, should agree as much as possible with the mathematical system of notation, and that those letters which have a fixed meaning for the mathematician, as e, i, d, l, π, Δ, Σ, &c., cannot be allowed to be employed with another meaning. We have tried to give to those expressions which are most frequently used a most simple symbol.

All expressions dependent on age are expressed as functions of the age.

The proposal is limited to those expressions most commonly in use.

§ 1. The age is denoted by a, $a+1$, $a+2$, . . . $a+h$; and the general symbol for a variable age is x, $x+1$, . . . $x+h$.

§ 2. The number of persons alive taken from the table of mortality is λ, as a function of the age $\lambda(a)\ \lambda(a+1)$ . . . $\lambda(x)\ \lambda(x+1)$ . . . (not in the form $\lambda_x\ \lambda_a$, where the age is annexed as an index). As the number of persons alive at any age is proportionate to the number of births to which the table refers, it is desirable to have an adequate form of notation. Be G the number of births, and $L(x)$ defined by the equation $G.L(x)=\lambda(x)$, then $L(x)$ is the number alive at the age x, supposing the number born to be 1. The number dying in a year is $\lambda(x)-\lambda(x+1)$, $L(x)-L(x+1)$, equal to $-\Delta\lambda(x)$ and $-\Delta L(x)$, also to be denoted by $\tau(x)$ and $T(x)$.

§ 3. Simple notations are recommended for the probabilities of living or dying in a certain time. They are—

$$W(x)=\frac{\lambda(x+1)}{\lambda(x)}=\frac{L(x+1)}{L(x)};\ W(x+1)=\frac{\lambda(x+2)}{\lambda(x+1)}=\frac{L(x+2)}{L(x+1)}$$

$$W^2(x)=W(x)\ W(x+1)=\frac{\lambda(x+2)}{\lambda(x)}=\frac{L(x+2)}{L(x)}$$

$$W^h(x)=W(x)\ W(x+1)\ W(x+2)\ldots W(x+h-1)=\frac{\lambda(x+h)}{\lambda(x)}=\frac{L(x+h)}{L(x)}$$

The probability of dying is denoted by w, defined by the equation

$$1=W^h(x)+w^h(x).$$

§ 4. The expectation of life exactly expressed by $\int_a^\infty \frac{L(x)dx}{L(a)}$, and approximately by $\frac{1}{2}+\frac{L(a+1)+L(a+2)+L(a+3)+\ .\ .\ .\ .}{L(a)}$ is denoted by $E(a)$.

§ 5. Unity with one year's interest added is denoted by r (say $r=1{,}03$); to avoid the negative exponents, unity discounted for one year is denoted by ρ equal to $\frac{1}{r}$.

§ 6. The present value of an annuity of 1 payable yearly, the first payment to be made directly, is denoted by R as a function of the age $R(a)$, $R(x)$ (the age not to be annexed as an index). If the first payment

of the annuity is deferred 1, 2,. . . n years, the function is denoted by ${}^1R(x)$, ${}^2R(x)$, . . . ${}^nR(x)$, the number of years deferred annexed at the top of the R on the left side. If the annuity is a temporary one for 1, 2, . . . t years, the function is $R_1(x)$, $R_2(x)$, . . . $R_t(x)$, the number of years annexed at the foot of the R, on the right side. Accordingly ${}^nR_t(x)$ is the value of a temporary annuity of t years duration, deferred n years. If the annuity is payable half-yearly, monthly, quarterly, $\frac{2}{2}$, $\frac{4}{4}$, $\frac{12}{12}$, is annexed at the top of the R, on the right side, $R^{\frac{2}{2}}(x)$, $R^{\frac{12}{12}}(x)$, ${}^nR^{\frac{4}{4}}(x)$, &c.

§ 7. The single premium (present value) for the assurance of the capital 1, payable at the end of the year when death occurs, is denoted by P, as a function of the age P(x); the annual premium payable for the whole term of life is $p(x)$. If the assurance is a temporary one for 1, 2, . . . t years only, or a deferred one (beginning in 1, 2, . . . n years), the function is denoted in conformity to § 6.

$$P_1(x),\ P_2(x),\ \ldots\ P_t(x),\ \text{ and }\ p_1(x),\ p_2(x),\ \ldots\ p_t(x)$$
$${}^1P(x),\ {}^2P(x),\ \ldots\ {}^nP(x),\ \text{ and }\ {}^1p(x),\ {}^2p(x),\ \ldots\ {}^np(x)$$

If the premium is payable half-yearly, quarterly, monthly, the annual premium (payable in 2, 4, 12 instalments), is $p^{\frac{2}{2}}(x)$, $p^{\frac{4}{4}}(x)$, $p^{\frac{12}{12}}(x)$. If the assurance is for the whole term of life, but the premium to be paid in m annual payments, the annual premium is denoted by ${}_{(m)}p(x)$, defined by the equation ${}_{(m)}p(x)=\dfrac{P(x)}{R_m(x)}$.

§ 8. The premium for the assurance of an endowment is denoted by German letters; the single premium, if the endowment is deferred n years, by ${}^n\mathfrak{P}(x)$, the annual premium by ${}^n\mathfrak{p}(x)$.

§ 9. The values of annuities for two joint lives aged a and $a+h$ are denoted by

$R(a,\ a+h)$ if the annuity is payable only during the joint duration of both lives,

$R\left(\dfrac{a}{a+h}\right)$ if the annuity is payable only after death of the person aged $a+h$ years, the person aged a being still alive.

§ 10. In conformity to § 9 the premium for the assurance of joint lives is denoted by

$P(a,\ a+h), p(a,\ a+h,$ assurance payable at the death of the first of both dying;

$P\left(\dfrac{a}{a+h}\right)$, $p\left(\dfrac{a}{a+h}\right)$, survivorship assurance payable at the death of the person aged $(a+h)$, the person aged a surviving.

Dr. Ph. Fischer, *Darmstadt.*
Dr. K. Heym, *Leipzig.*
G. Hopf, *Gotha.*
Wilhelm Lazarus, *Hamburg.*
Dr. Laudi, *Trieste.*
Prof. Weninger, *Pesth.*
Dr. A. Wiegand, *Halle.*
Prof. Dr. Wittstein, *Hanover.*
Dr. A. Zillmer, *Stettin.*

October, 1863.

Remarks.

At § 2. It ought to be mentioned that all numbers in tables of mortality are proportionate numbers only, and that it might have been sufficient in the notations to have had regard to this peculiarity. But it was difficult to determine a fixed point in the tables of mortality, which do not always begin at birth, and therefore it has been considered advisable to introduce both notations, $\lambda(x)=\mathrm{GL}(x)$ and $\tau(x)=\mathrm{GT}(x)$. That in all cases where the numbers in the table of mortality are used to determine a probability, and appear in the numerator as well as in the denominator, it need scarcely to be mentioned, it is optional to use λ or L, τ or T.

At § 3. One year has been adopted as the unit of time in the probabilities of living or dying during a fixed period. The same form of notation is used for shorter periods than a year; thus, $\mathrm{W}^{\frac{1}{2}}(x)$ denotes the probability for a person aged x years living six months, $w^{\frac{1}{\infty}}$ the probability of dying in the next moment.

At § 6. It would be very desirable to have a short expression in words for the value of an annuity of 1 payable during the whole term of life annually, the first payment just due. The expression, "Mise," used by many authors, is recommended.

It will not be uninteresting to see some formulæ expressed by the new symbols:—

$$\mathrm{R}_1(x)=1;\ {}^1\mathrm{R}_1(x)=\mathrm{W}(x)\rho;\ {}^2\mathrm{R}_1(x)=\mathrm{W}^2(x)\rho^2;\ {}^3\mathrm{R}_1(x)=\mathrm{W}^3(x)\rho^3\ \ldots$$

$$\mathrm{R}(x)=\frac{\mathrm{L}(x)\rho^x+\mathrm{L}(x+1)\rho^{x+1}+\mathrm{L}(x+2)\rho^{x+2}+\ \ldots}{\mathrm{L}(x)\rho^x}$$

$$\mathrm{R}(x)=\mathrm{R}_1(x)+{}^1\mathrm{R}_1(x)+{}^2\mathrm{R}_1(x)+{}^3\mathrm{R}_1(x)+\ \ldots$$

$$\mathrm{R}_t(x)=\mathrm{R}_1(x)+{}^1\mathrm{R}_1(x)+{}^2\mathrm{R}_1(x)+\ \ldots\ {}^{t-1}\mathrm{R}_1(x)$$

$${}^t\mathrm{R}(x)={}^t\mathrm{R}_1(x)+{}^{t+1}\mathrm{R}_1(x)+{}^{t+2}\mathrm{R}_1(x)+\ \ldots\ =\mathrm{W}^t(x)\rho^t\mathrm{R}(x+t)$$

$$\mathrm{R}(x)=\mathrm{R}_t(x)+{}^t\mathrm{R}(x).$$

If the annuity is payable momently,

$$\mathrm{R}^{\frac{\infty}{\infty}}(x)=\int_a^{\infty}\frac{\mathrm{L}(x)\,\rho^x d^x}{\mathrm{L}(a)\rho^a}.$$

At § 7. From the well-known equation

$${}^1\mathrm{R}(x)+\left(1+\frac{1}{r-1}\mathrm{P}(x)\right)=\frac{1}{r-1},$$

it follows

$$\mathrm{P}(x)=\rho-(1-\rho)^1\mathrm{R}(x)=1-(1-\rho)\mathrm{R}(x)$$

$$p(x)=\frac{\mathrm{P}(x)}{\mathrm{R}(x)}\quad {}^n p(x)=\frac{{}^n\mathrm{P}(x)}{{}^n\mathrm{R}(x)}\quad p_t(x)=\frac{\mathrm{P}_t(x)}{\mathrm{R}_t(x)}\quad {}_{(m)}p(x)=\frac{\mathrm{P}(x)}{\mathrm{R}_m(x)}.$$

$\mathrm{P}(x+n)-p(x)\mathrm{R}(x+n)$ is the surrender value of a policy for an assurance after n year's duration, of which the annual net premium is $p(x)$ when the $(n+1)$th premium is just due but unpaid, $\mathrm{P}(x+n)-p(x)^1\mathrm{R}(x+n)$ if the $(n+1)$th premium has just been paid.

$\mathrm{P}(x+n)$ being equal to $p(x+n)\,\mathrm{R}\,(x+n)$, this value becomes $\{p(x+n)-p(x)\}\,\mathrm{R}\,(x+n)$ and $\{p(x+n)-p(x)\}\,\mathrm{R}\,(x+n)+p\,(x)$, or,

according to the notation introduced in mathematics by Moigno, $R(x+n)/_{x}^{x+n}p(x)$ and $p(x)+R(x+n)/_{x}^{x+n}p(x)$.

This value changes if the next payment is due in the time t; it changes if an annual premium only for m years is charged instead of a premium for the whole term of life. It is evident that the value of a policy depends on so many different circumstances that they cannot be expressed in one single symbol. But perhaps it is convenient to express this value as a function of the net premium (symbol of the function, ϕ), adding the time elapsed since the policy was taken at top left hand, and the time till the next payment of premium at the foot right hand. This last addition ceases when either no further payment of premium is required, or when the next payment is due immediately, viz.:—

$$^{n}\phi\{p(x)\}=P(x+n)-p(x)R(x+n)$$

$$^{n}\phi_{1}\{p(x)\}=P(x+n)-p(x)^{1}R(x+n)$$

$$^{n}\phi_{t}\{p(x)\};\ ^{n}\phi\{_{(m)}p(x)\};\ ^{n}\phi\{P(x)\}=P(x+n)$$

$$^{n}\phi\{P_{t}(x)\}=P_{(t-n)}(x+n);\ ^{n}\phi\{^{m}\mathfrak{P}(x)\}=\,^{m-n}\mathfrak{P}(x+n);\ ^{n}\phi\{R(x)\};\ ^{n}\phi\{^{m}\mathfrak{p}(x)\}\ \&c.$$

At § 8. $^{n}\mathfrak{P}(x)=W^{n}(x)\rho^{n}=\,^{n}R_{1}(x)$

$$^{n}\mathfrak{p}(x)=\frac{^{n}\mathfrak{P}(x)}{R_{n}(x)}.$$

At § 9. As a symbol for the value of an annuity for two persons payable till both have died, the age being a and $a+h$, $R(a//a+h)=R(a)+R(a+h)-R(a,\ a+h)$ is recommended.

WILHELM LAZARUS.

THE DEMONSTRATION OF CERTAIN FORMULÆ.

To the Editor of the Assurance Magazine.

Sir,—I beg to submit demonstrations of several of the formulæ for whole life, temporary, deferred, and endowment assurances.

Let (a) denote the value of an annuity of £1 on the joint lives of the last v survivors of the lives m, m_1, m_2, &c.; and (A) the value of an assurance of £1 on the same lives.

The present value of a perpetuity of £1 per annum, the first payment due immediately, is made up of the present value of an annuity of £1 during the continuance of a given status, and the present value of a reversionary annuity of £1 to commence at the end of the year in which the status ceases, which latter annuity is equivalent to the present value of an assurance of $£1+\frac{1}{i}$. Now, the reversionary annuity of £1 can be secured by an assurance of $£1+\frac{1}{i}$ during a certain term, together with an endowment of the same sum payable at the end of that term.

Hence, the value of a perpetuity of £1, first payment immediate, equals the present value of a temporary annuity for t years on the joint lives of the last v survivors + present value of an assurance for t years of $£1+\frac{1}{i}$ on the failure of the joint existence of the last v survivors + present value of

an endowment of £$1+\frac{1}{i}$ payable at the end of t years, in the event of the last v survivors being alive at that time; or,

$$1+\frac{1}{i}=1+(a)_{\overline{t-1}|}+\left\{(\mathrm{A})_{\overline{t}|}+r^t p_{\overline{(m,\,m_1,\,m_2,\,\&c.)t}}^{\;v}\right\}\overline{1+\frac{1}{i}}$$

$$\therefore\ 1=d(1+(a)_{\overline{t-1}|})+(\mathrm{A})_{\overline{t}|}+r^t p_{\overline{(m,\,m_1,\,m_2,\,\&c.)t}}^{\;v},\ \text{since } d=\frac{i}{1+i}.$$

$$\therefore\ (\mathrm{A})_{\overline{t}|}=1-d(1+(a)_{\overline{t-1}|}-r^t p_{\overline{(m,\,m_1,\,m_2,\,\&c.)t}}^{\;v}\quad .\ .\ .\ .\ .\quad (1)$$

which is the ordinary formula for the present value of a temporary assurance for t years.

Increase t without limit, the present value of the endowment vanishes, and we have

$$(\mathrm{A})=1-d(1+(a))\quad .\ .\ .\ .\ .\ .\ .\quad (2)$$

the ordinary formula for the present value of a whole life assurance.

The annual premium in the two cases is at once obtained by dividing (1) by $1+(a)_{\overline{t-1}|}$, which gives

$$\frac{1-r^t p_{\overline{(m,\,m_1,\,m_2,\,\&c.)t}}^{\;v}}{1+(a)_{\overline{t-1}|}}-a\quad .\ .\ .\ .\ .\ .\ .\quad (3)$$

for the annual premium of a temporary assurance for t years, and dividing (2) by $1+(a)$, which gives

$$\frac{1}{1+(a)}-d,\quad .\ .\ .\ .\ .\ .\ .\ .\quad (4)$$

for the annual premium of a whole life assurance.

From (1) it is at once apparent, that the present value of an endowment assurance of £1, to be paid on the failure of the joint lives of the last v survivors within t years, or at the expiration of that time, should they be then alive, is

$$(\mathrm{A})_{\overline{t}|}+r^t p_{\overline{(m,\,m_1,\,m_2,\,\&c.)t}}^{\;v}=1-d(1+(a)_{\overline{t-1}|})\quad .\ .\ .\ .\quad (5)$$

The annual premium is obtained by dividing (5) by $1+(a)_{\overline{t-1}|}$, which gives

$$\frac{1}{1+(a)_{\overline{t-1}|}}-d\quad .\ .\ .\ .\ .\ .\ .\quad (6)$$

The simple rule then for the annual premium of an endowment assurance is:—Take the reciprocal of the present value of an annuity of £1 for t years, payable in the beginning of each year, less the discount of £1 for one year.

This rule will be found extremely useful, especially in cases where more than one life is involved.

The corresponding commutation formulæ are—

Single premium for an endowment assurance on a single life,

$$=1-d\frac{\mathrm{N}_{m-1}-\mathrm{N}_{m+t-1}}{\mathrm{D}_m};$$

Single premium for a similar assurance on two lives,

$$=1-d\frac{N_{m-1,\,m_1-1}-N_{m+t-1,\,m_1+t-1}}{D_{m,\,m_1}};$$

Annual premium for an endowment assurance on a single life,

$$=\frac{D_m}{N_{m-1}-N_{m+t-1}}-d;$$

Annual premium for a similar assurance on two lives,

$$=\frac{D_{m,\,m_1}}{N_{m-1,\,m_1-1}-N_{m+t-1,\,m_1+t-1}}-d.$$

Again, the present value of a perpetuity of £1 deferred t years, subject to the last v survivors being then alive, equals the present value of an annuity of £1 deferred t years to continue during the joint lives of the last v survivors + the present value of an assurance of £1 $+\frac{1}{i}$ deferred t years and payable on the failure of the joint existence of the last v survivors. Hence,

$$r^t p_{\overline{(m,\,m_1,\,m_2,\,\&c.)t}}^{\;v}\frac{1}{i}=(a)_{\rceil t}+(A)_{\rceil t}\left(1+\frac{1}{i}\right)$$

$$\therefore A_{\rceil t}=r^{t+1}p_{\overline{(m,\,m_1,\,m_2,\,\&c.)t}}^{\;v}-d(a)_{\rceil t} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and for the annual premium (7) has to be divided by $1+(a)_{\overline{t-1}\rceil}$.

I remain, Sir,

Your obedient servant,

MARCUS N. ADLER.

London, 15*th March*, 1864.

ON THE PAYMENT OF $\frac{1}{m}$ YEARLY PREMIUMS.

To the Editor of the Assurance Magazine.

Sir,—I find in No. 54 of the *Assurance Magazine* Mr. Laundy's method of obtaining half-yearly and quarterly premiums from the annual premium. He derives it very skilfully from Mr. Orchard's expression given in the introduction of his valuable work, *Single and Annual Assurance Premiums.* I am much surprised to see Mr. Laundy availing himself of this expression, since another is nearer and more directly to be got at. Having, however, not found it till now in any work, it might prove useful to publish it here for general use.

Putting, for the age of x years,

π as the annual premium on a premium paid half-yearly,
p " " " " annually,
$a'^{\frac{2}{2}}$ as the annuity to be paid half-yearly in advance,
a' " " " annually "

we find

$$\pi a'^{\frac{2}{2}}=pa' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Substituting $a'^{\frac{1}{2}}=a'-\frac{1}{4}$, (1) becomes

$$\pi(a'-\tfrac{1}{4})=pa' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2);$$

thus we get

$$\pi=\frac{pa'}{a'-\frac{1}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Substituting $a'=1+a$, (3) becomes

$$\pi=p\cdot\frac{1+a}{1+a-\frac{1}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\pi=p\left(1+\frac{0{\cdot}25}{0{\cdot}75+a}\right) \quad . \quad . \quad . \quad . \quad . \quad (5).$$

This is Laundy's expression.

To find the annual premium on the payment of $\frac{1}{m}$ yearly premiums, we start from the approximate expression

$$a'^{\frac{1}{m}}=a'-\frac{m-1}{2m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Thus we have

$$\pi a'^{\frac{1}{m}}=pa',$$

$$\pi=p\cdot\frac{a'}{a'-\frac{m-1}{2m}}$$

$$=\frac{1+a}{1+a-\frac{m-1}{2m}};$$

or, after a simple operation,

$$\pi=p\cdot\left(1+\frac{\frac{m-1}{2m}}{\frac{m+1}{2m}+a}\right) \quad . \quad . \quad . \quad . \quad . \quad (7).$$

This is Laundy's general expression.

I am, Sir,

Yours most obediently,

DR. AUGUST WIEGAND,

Director of Life Assurance Society, Iduna.

Halle, Germany, Prussia,
May 14, 1864.

ON THE SAME SUBJECT.

To the Editor of the Assurance Magazine.

SIR,—Having been allowed the privilege of perusing the preceding letter of Dr. August Wiegand, I can but express my gratification that a subject of such minor importance should have attracted the attention of your distinguished correspondent: and beg to thank him, through the

medium of your *Journal*, for supplementing, as he has done, the demonstration I gave upon this subject in my letter which appeared in vol. xi., p. 232. The value of the Doctor's remarks will be duly appreciated by your readers.

It is quite true that I might have deduced the expression (5) in the Doctor's letter by the "nearer and more direct" method which he lays down; but had I done so I should have failed to show that which I intended and expressed—Mr. Orchard's quantity c_x in terms of the annual premium (π_x), viz.:—

$$\pi_x \cdot \frac{\cdot 25}{\cdot 75+a_x}=c_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

that is, the increment to be made to the annual premium when paid half-yearly.

Having, then, had in view the two objects, first, of deducing, as above stated, an expression for c_x in terms of π_x, and secondly, another for the value of $\frac{1}{2}(\pi_x+c_x)$, I obtained at once the following obvious equation:—

$$\tfrac{1}{2}\left(\pi_x+\pi_x \cdot \frac{\cdot 25}{\cdot 75+a_x}\right)=\frac{\pi_x}{2}\left(1+\frac{\cdot 25}{\cdot 75+a_x}\right) \quad . \quad . \quad (b)$$

Hence it will appear that my process furnishes, as I proposed, both the means of obtaining the increment (a) to be made to the annual premium when payable half-yearly, and by (b) of finding, by a method as simple in the one case as the other, the half-yearly premium itself by direct calculation. I submit, therefore, that the expression I gave could not, under the circumstances, well be derived in a much nearer or more direct way.

Allow me to add, that when I before addressed you upon this subject, I intended only to submit what appeared to me to be a convenient and simple *arithmetical* process for forming a table for passing from the yearly premium to the equivalent premium when paid half-yearly, quarterly, or otherwise; and I think that the table appended to my letter, computed by that process, was produced with, perhaps, about as small an amount of labour as possible.

Another method of arriving at the same results, which I now beg to submit, may not be deemed unworthy of being added to my previous communication.

By eliminating the value a_x and solving in terms of π_x, more symmetry will be imparted to the expression, and we further obtain a constant for each rate of interest as one of the terms of the denominator. I am indebted to Mr. Samuel Younger, one of your talented contributors, for the suggestion, and avail myself of an extract from a communication from that gentleman upon this subject, in which he thus treats the case:—

"The half-yearly premium being H_x, we have

$$H_x=\frac{1-d(1+a_x)}{2(a_x+\cdot 75)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

"Now, from $\pi_x=\dfrac{1}{1+a_x}-d$, we find

$$1+a_x=\frac{1}{\pi_x+d} \text{ and } \cdot 75+a_x=\frac{1}{\pi_x+d}-\cdot 25.$$

"Substituting these values of $1+a_x$ and $\cdot 75+a_x$ in (1), we get

$$H_x = \frac{1-\dfrac{d}{\pi_x+d}}{2\left(\dfrac{1}{\pi_x+d}-\cdot 25\right)} = \tfrac{1}{2}\,\frac{\pi_x}{1-\cdot 25(\pi_x+d)}."$$

This latter equation may be again reduced to

$$\frac{\pi_x}{2-\frac{1}{2}d-\frac{1}{2}\pi_x},$$

which, when the rate of interest is 3 per cent., becomes

$$H_x = \frac{\pi_x}{1\cdot 9854-\frac{1}{2}\pi_x},$$

which is extremely simple.

The example worked in my last letter will, by this formula, be as follows:—

$$\pi_x = \cdot 0410 \;\therefore\; 1\cdot 9854 - \cdot 0205 = 1\cdot 9649,$$
and $\cdot 0410 \div 1\cdot 9649 = \cdot 02087$, as before.

The general formula will be found readily from the foregoing to be

$$\frac{1}{m}\cdot\frac{\pi_x}{1-\dfrac{m-1}{2m}(\pi_x+d)}.$$

Apologizing for the length of this communication,

I am, Sir,

Your obedient servant,

Eagle Insurance Company, SAMUEL L. LAUNDY.
6th September, 1864.

ON A PARTICULAR ARRANGEMENT OF ELEMENTARY VALUES.

To the Editor of the Assurance Magazine.

SIR,—1. The values of annuities and assurances of all kinds consist of certain elements variously combined. These elements are not usually exhibited in detail, their combinations being otherwise attainable.

2. But an intimate knowledge of details will enable its possessor to surmount such difficulties as occur in the treatment of complex questions involving many lives. Something may also be gained by particular arrangements of elementary values. For these reasons the following brief exposition is offered.

3. Our elementary table contains the logarithms, for the Carlisle life table, from age 90 upwards, of the following quantities:—

l_x = number that complete age x;
d_x = number that die in the $(x+1)$th year of age;
v = an unit of money, discounted for one year (3 per cent.);

and combinations of these, as shown in the headings.

4. The present value of £1, to be received in event of a life now aged x completing the $(x+1)$th year of age, is $\frac{vl_{x+1}}{l_x}=vp_x$. This is an endowment.

The logarithms of a series of deferred endowments may be formed by continuous addition of $\log vp_x$, from the youngest age upwards.

5. The present value of £1 to be received at the end of the year, provided that a life aged x fail within the $(x+1)$th year, is $\frac{vd_x}{l_x}=vr_x$. This is an assurance for one year. If $\frac{d_{x+1}}{d_x}=q_x$, then the logarithms of a series of deferred assurances for one year may be formed by continuous addition of $\log vq_x$ to the initial value of $\log vr_x$.

6. The present value of £1 to be received in event of two lives x and y jointly surviving one year, is $\frac{vl_{(x.y)}}{l_{(x.y)}}=vp_{x.y}$; and the present value of £1 to be received at the end of the year, provided that one or both lives fail within the next year, is $\frac{vd_{x.y}}{l_{x.y}}=vr_{x.y}$. These symbols are under the same laws as those for single lives.

7. The present value of £1 to be received at the end of the year, provided that a life x fails in the $(x+1)$th year, and that a life y survives the instant at which x dies, is $\frac{vd_xl_{y+\frac{1}{2}}}{l_{x.y}}=vr_xp_{y\frac{1}{2}}$. This is a survivorship assurance for one year. Taking this as an initial value, we may, by adding to its logarithm continuously $\log vq_x+\log p_{y+\frac{1}{2}}$, obtain those of a deferred series.

8. It thus appears that the construction of the logarithmic values of the foregoing elements is uniform. But not only this: there is a relation which connects them. When the logarithms of successive deferred endowments have been formed, then, by adding to these the corresponding logarithms of a series $\frac{d_x}{l_{x+1}}=S_x$, we may derive those of deferred yearly assurances; and if to these again we add corresponding terms of the series $\log\frac{l_{y+n+\frac{1}{2}}}{l_y}$, we obtain the logarithms of deferred yearly survivorship assurances on x against y. This will be clearly seen in the following arrangement. Carlisle 3 per cent. $y=90$.

x.	1. Endowments. $\Sigma(\log vp_x)$.	2. $\text{Log}\ \frac{d_x}{l_{x+1}}=S_x$.	3. Assurances. (1)+(2).	4. $\text{Log}\ \frac{l_{y+n+\frac{1}{2}}}{l_y}$.	5. Survivorship Assurances. (3)+(4). x against y.
95	9·87177	9·48337	9·35514	9·93938	9·29452
6	·75248	·44370	·19618	·80195	·99813
7	·63050	·45593	·08643	·65727	·74370
8	·51292	·43573	·94865	·51981	·46846
9	·41294	·34679	·75973	·39178	·15151
100	·29095	·45593	·74688	·27096	·01784
1	9·13199	·60206	·73405	·15947	·89352
2	8·89730	9·82391	·72121	·05183	·77304
3	·40734	0·30103	·70837	·94462	·65299
4	..	..	·39451	·84771	·24222

1. This is the sum of $\log vp_x$ from age x (inclusive) upwards.
2. This is $\log S_x$, from the elementary table.
3. This is the sum of the two preceding columns. See para. 5.
4. In this the initial value is $\log p_{y\frac{1}{2}}$, and the successive differences are $\log p_{y+\frac{1}{2}}$. These two symbols must be carefully distinguished. One is the probability of living half a year, the other that of living from the middle of one year to the middle of the next year.
5. This is the sum of the two preceding columns. See para. 7.

9. If the numbers corresponding to columns 1, 3, and 5, be summed from the bottom upwards, they will possess all the properties of the symbols N, M, and $M_{\overset{1}{\overline{x.y}}}$ in commutation tables. The writer does not mean to propose this as a model arrangement, yet for ease of calculation it seems to be very convenient. What he does propose is to sum the numerical values from the youngest age downwards; and he conceives that by this arrangement some remarkable advantages would be gained. For illustration of this, the following specimen table may suffice. It exhibits assurances, ages 47 and 52; Carlisle 3 per cent.

Assurances. $x=52$; $y=47$. *Diff.* 5 *years.*

Term of Assurance. Years. n.	(1) A_x.	(2) A_y.	(3) $A_{\overset{1}{\overline{x.y}}}$.	(4). $A_{\overline{x.y}}^{\ 1}$.
1	·01476	·01418	·01465	·01407
2	·02975	·02712	·02932	·02671
3	·04473	·03929	·04378	·03840
4	·05990	·05071	·05822	·04919
5	·07523	·06237	·07261	·05999
6	·09129	·07424	·08746	·07076
7	·10897	·08629	·10356	·08146
8	·12854	·09833	·12108	·09187
9	·15041	·11053	·14032	·10209
10	·17234	·12285	·15925	·11206
11	·19379	·13576	·17741	·12212
12	·21429	·14998	·19437	·13278
13	·23420	·16571	·21041	·14411
14	·25337	·18329	·22538	·15626
15	·27184	·20092	·23930	·16794
16	·28976	·21817	·25232	·17887
17	·30716	·23466	·26448	·18884
18	·32420	·25066	·27592	·19806
19	·34074	·26607	·28658	·20649
20	·35809	·28092	·29729	·21416
21	·37644	·25933	·30813	·22114
22	·39548	·30932	·31886	·22742
23	·41515	·32301	·32942	·23305
24	·43356	·33631	·33880	·23801
25	·45098	·35026	·34719	·24270
Life	·57599	·52544	·38344	·26834

$$A_{x.y}=(3)+(4).\quad A_{\overset{2}{\overline{x.y}}}=(1)-(3);\ A_{\overline{x.y}}^{\ 2}=(2)-(4).$$

$$A_{\overline{x.y}}^{2\,2}=(1)+(2)-\{(3)+(4)\}.$$

Deferred assurances are obtained by subtracting the temporary from the whole life values.

10. This table scarcely requires explanation. The value of an assurance on the joint lives 47 and 52 is ·38344+·26834=·65178. An assurance on this status for ten years is ·15925+·11206=·27131, and the difference of these, =·38047, is an assurance on 47 and 52 deferred for 10 years. An assurance on the last survivor of lives now aged 47 and 52 would be ·57599+·52544—·38344—·26834=·44965. In like manner, any other conditions within the limits tabulated may be determined by simple addition and subtraction.

11. A complete set of such tables would be exhaustive. But the uses of them lie within moderate limits. Up to $n=25$, and $x-y$ or $y-x=10$, they have already been computed, and the writer would willingly complete them to such extent as may be thought needful.

12. It is possible, with small trouble and with sufficient exactness, to pass from the elementary numerical values at one rate of interest to those at another rate. The following is an example of this. The values are those of survivorship assurances, 30 against 35.

n	3 per Cent.		Subtract.		4 per Cent.	
	Single Years.	Temporary.	S	t	Single Years.	Temporary.
1	·00976	·00976	9	9	·00967	·00967
2	·00938	·01914	18	27	·00920	·01887
3	·00885	·02799	25	52	·00860	·02747
4	·00834	·03633	32	84	·00802	·03549
5	·00801	·04434	38	122	·00763	·04312
6	·00768	·05202	43	165	·00725	·05037
7	·00749	·05951	49	214	·00700	·05737
8	·00729	·06680	54	268	·00675	·06412
9	·00710	·07390	59	327	·00651	·07063
10	·00715	·08105	66	393	·00649	·07712

Here the column marked S is a quantity depending on the single year values, and t is the sum of that column. These being taken from the 3 per cent. columns give the 4 per cent. columns. S is taken from a small subsidiary table. In like manner, we may proceed from 4 to 5 per cent., and so on. The work is very much less than if logarithms were used.

I am, Sir,

Yours truly,

W. H. OAKES,
Lieut.-Col.

London, August, 1864.

THE

ASSURANCE MAGAZINE,

AND

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

Solutions of General Problems in Survivorships. *By* WILLIAM MATTHEW MAKEHAM, *Fellow of the Institute of Actuaries.*

[Read before the Institute, 28th November, 1864.]

THE following collection of problems is intended to serve two purposes—first, as a continuation of a former article, published in vol. x. of this *Magazine,* under the title of "Solutions of the Compound-Survivorship Assurance Problems," which treats of cases involving three lives only; and secondly, as an introduction, or preliminary step, to a complete exposition of the writer's method of constructing mortality and annuity tables upon Mr. Gompertz's celebrated hypothesis, a brief account of which method will be found in the eighth and ninth volumes.

In the solutions of the three-life cases, it was shown that all the compound-survivorship problems, which had been approximatively solved by Baily and Milne, are reducible to the compound-survivorship annuity ${}_{\underset{\mathrm{I}}{\urcorner}\mathrm{BC}}\mathrm{A}$ (combined with simple-survivorship reversions), and consequently that the construction of complete tables of this function will enable us to effect the numerical solution of the problems in question. In the exposition above referred to, a method of constructing these tables will be given, by which the labour is very much reduced; but as even by this method the work of construction is by no means inconsiderable, it may not be considered useless to ascertain to what extent such tables when

constructed will be serviceable—that is, what other problems they will enable us to solve in addition to those already investigated.

The conclusion derived from an examination of the following solutions is, that *all* cases involving not more than *four* lives are within our power, by the construction of complete tables of $\rceil^{\mathrm{BC}}_{\mathrm{I}}\mathrm{A}$, $\rceil^{\mathrm{BCD}}_{\mathrm{I}}\mathrm{A}$, and $\rceil^{\mathrm{CD}}_{\mathrm{I}}\mathrm{AB}$. This is shown by the solutions of Problems IV. and XII., which determine the values of $\rceil^{\overline{\mathrm{B\,C\,D}}}_{\mathrm{III\,II\,I}}\mathrm{A}$ and $\underset{\mathrm{IV}}{\mathfrak{A}}\underset{\mathrm{III}}{\mathbb{B}}\underset{\mathrm{II}}{\mathbb{C}}\underset{\mathrm{I}}{\mathbb{D}}$, where each of the three lives which are to fail before A must die in a given order. The investigation is confined to these two problems and those which lead to them, it being no part of the writer's plan to give a complete collection of the numerous cases which might be formed by combining two or more of these together.

There are three fundamental problems upon which the solution of the others depends, viz., the first, the fifth, and the ninth. The exact solution of these fundamental problems varies according to the nature of the function expressing the law of mortality—which is not the case with any of the others, except so far as they are dependent upon the fundamental problems. The solutions given of these fundamental problems are approximations only, although it is believed very close ones; but *exact* solutions of these cases will be given in treating of the law of mortality, which will enable us to test the approximations now given.

The very elegant and convenient expression for the value of the annuity $\rceil^{\mathrm{BC}}_{\mathrm{I}}\mathrm{A}$, viz., $\mathrm{S}\,\dfrac{{}^{n-1}b-{}^{n}b}{b}\cdot\dfrac{{}^{n-\frac{1}{2}}c}{c}\cdot\rceil^{n-1}\mathrm{A}$ (which denotes an annuity on *A*'s life deferred until the death of *B* during *C*'s lifetime), was, it appears, first given to the world by Mr. Griffith Davies. The principle of this solution is a very important one—it has been adopted in the solution of Problem IX; a deferred survivorship assurance, on *A*'s life against *B*'s, being substituted for a deferred annuity on the life of *A*.

For the purpose of generalising the solutions, I have extended the signification of the notation used by Mr. Milne. Thus, the characters A and $\mathfrak{A}$ denote annuities and assurances on the joint existence of any given number of lives, A_1, A_2, A_3, . . . while a denotes the product of the numbers living at the several ages of these lives. The characters ${}^{\frac{1}{2}}\mathrm{A}$, ${}^{\frac{1}{2}}\mathfrak{A}$, ${}^{\frac{1}{2}}a$, and $\mathrm{A}_{\frac{1}{2}}$, $\mathfrak{A}_{\frac{1}{2}}$, $a_{\frac{1}{2}}$, signify the same on lives each a half year older and younger respectively. The adoption of Mr. Milne's system of notation has prevented the use of the "commutation" method, by which many of the formulæ

would have been rendered much more concise and elegant, but anyone who may take the trouble to peruse this article will see where that method may be substituted with advantage.

Problem I.

To determine ($\neg_{\overset{\mathrm{BC}}{\mathrm{I}}}\mathrm{A}$) the present value of an annuity on the joint lives $A_1, A_2, A_3, \ldots$ after the failure of the joint existence of the lives B_1, B_2, B_3, provided that event shall happen during the joint existence of the lives C_1, C_2, C_3.

Solution.

The contingent benefit, of which the value is to be determined, is, in fact, a deferred annuity on the joint lives (A), the first payment of which is not to be made until the end of the year in which the joint lives (B) shall fail during the joint existence of the lives (C). If we have tables showing the number of the combinations (similar to each of these sets) which survive each year of age, and the decrements thereof, we may approximate with sufficient accuracy to the chance of the joint lives (C) surviving the joint lives (B) in the nth year by the formula $\frac{{}^{n-1}b - {}^{n}b}{b} \cdot \frac{{}^{n-1}c + {}^{n}c}{2c}$. In the case of a single life in each set, this expression is exact upon the hypothesis of a uniform distribution of the deaths of each year; but in cases involving a plurality of lives in each set, it will be found abundantly accurate for all practical purposes.

The value of an annuity on the joint lives (A) deferred $n-1$ years, or $\neg_{n-1}\mathrm{A}$, is $\frac{{}^{n-1}a}{a} \cdot v^{n-1} \cdot {}^{n-1}\mathrm{A}$. Multiplying this expression into the former, we find the value of the contingency due to the nth year to be $\frac{{}^{n-1}b - {}^{n}b}{b} \cdot \frac{{}^{n-1}c + {}^{n}c}{2c} \cdot \frac{{}^{n-1}av^{n-1}}{a} \cdot {}^{n-1}\mathrm{A}$, and using S as the symbol of summation, commencing with $n=1$, we have

$$\neg_{\overset{\mathrm{BC}}{\mathrm{I}}}\mathrm{A} = S\, \frac{{}^{n-1}b - {}^{n}b}{b} \cdot \frac{{}^{n-1}c + {}^{n}c}{2c} \cdot \frac{{}^{n-1}av^{n-1}}{a} \cdot {}^{n-1}\mathrm{A} \quad . \quad . \quad . \quad (1)$$

When applied to a single life in each set, this formula is in substance the same as that given by Mr. Griffith Davies, at p. 356 of his work on life contingencies.

Adapting the formula to the case in which the number of lives in each set is the same as before, but each life one year older, we have

F 2

$$\urcorner_{1}^{{}^1\mathrm{B}{}^1\mathrm{C}}{}^1\mathrm{A}=S\frac{{}^nb-{}^{n+1}b}{{}^1b}\cdot\frac{{}^nc+{}^{n+1}c}{2\,{}^1c}\cdot\frac{{}^nav^{n-1}}{{}^1a}\cdot{}^n\mathrm{A}$$

$$=\frac{(abc)}{{}^1(abc)v}\cdot S\frac{{}^nb-{}^{n+1}b}{b}\cdot\frac{{}^nc+{}^{n+1}c}{2c}\cdot\frac{{}^nav^n}{a}\cdot{}^n\mathrm{A},$$

where the expression affected by the symbol of summation is evidently $\urcorner_{1}^{\mathrm{BC}}\mathrm{A}$ minus its first term $\frac{1}{2}\left(1-\frac{{}^1b}{b}\right)\left(1+\frac{{}^1c}{c}\right)\mathrm{A}$. Therefore we have

$$\frac{{}^1(abc)v}{(abc)}\cdot\urcorner_{1}^{{}^1\mathrm{B}{}^1\mathrm{C}}{}^1\mathrm{A}=\urcorner_{1}^{\mathrm{BC}}\mathrm{A}-\tfrac{1}{2}\left(1-\frac{{}^1b}{b}\right)\left(1+\frac{{}^1c}{c}\right)\mathrm{A},$$

and

$$\urcorner_{1}^{\mathrm{BC}}\mathrm{A}=\tfrac{1}{2}\left(1-\frac{{}^1b}{b}\right)\left(1+\frac{{}^1c}{c}\right)\mathrm{A}+\frac{{}^1(abc)v}{(abc)}\cdot\urcorner_{1}^{{}^1\mathrm{B}{}^1\mathrm{C}}{}^1\mathrm{A}\quad . \quad . \quad (2),$$

by which formula the value of the annuity $\urcorner_{1}^{\mathrm{BC}}\mathrm{A}$ may be derived from the value of a similar annuity where all the lives involved are one year older.

Returning to the formula (1) we get, by multiplying out,

$$\urcorner_{1}^{\mathrm{BC}}\mathrm{A}=S.\frac{{}^{n-1}(bc)-{}^n(bc)+{}^{n-1}b.{}^nc-{}^nb.{}^{n-1}c}{2(bc)}\cdot\frac{{}^{n-1}a.v^{n-1}}{a}\cdot{}^{n-1}\mathrm{A}$$

$$=\tfrac{1}{2}S.\left(\frac{{}^{n-1}(bc)-{}^n(bc)}{(bc)}-\frac{{}^{n-1}({}^1bc)}{({}^1bc)}\cdot\frac{{}^1b}{b}+\frac{{}^{n-1}(b{}^1c)}{(b{}^1c)}\cdot\frac{{}^1c}{c}\right)\cdot\frac{{}^{n-1}a.v^{n-1}}{a}\cdot{}^{n-1}\mathrm{A}$$

$$=\tfrac{1}{2}\left[S.\frac{{}^{n-1}(bc)-{}^n(bc)}{(bc)}\cdot\frac{{}^{n-1}a.v^{n-1}}{a}\cdot{}^{n-1}\mathrm{A}-\right.$$

$$\left.S\left(\frac{{}^{n-1}({}^1bc)}{({}^1bc)}\cdot\frac{{}^1b}{b}-\frac{{}^{n-1}(b{}^1c)}{(b{}^1c)}\cdot\frac{{}^1c}{c}\right)\cdot\frac{{}^{n-1}a.v^{n-1}}{a}\cdot{}^{n-1}\mathrm{A}\right].$$

The first term within the brackets [ ] denotes the value of a deferred annuity on the joint lives (*A*), the first payment of which is to be made at the end of the year in which the joint lives (*BC*) shall fail; in other words, a reversionary annuity on the joint lives (*A*) after the joint lives (*BC*), the value of which may be shown to be A—ABC. The remaining terms resolve themselves into the values of annuities of a novel and somewhat remarkable character, the form of which is

$$S\cdot\frac{{}^{n-1}(bc)}{(bc)}\cdot\frac{{}^{n-1}a.v^{n-1}}{a}\cdot{}^{n-1}\mathrm{A},$$

or

$$\mathrm{A}+\frac{{}^1(bc)}{(bc)}\cdot\urcorner^{1}\mathrm{A}+\frac{{}^2(bc)}{(bc)}\urcorner^{2}\mathrm{A}+\frac{{}^3(bc)}{(bc)}\urcorner^{3}\mathrm{A}+\ldots.$$

which it may readily be perceived is the value of an annuity on the

joint lives A, beginning with £1, and increasing by £1 every year during the joint existence of the lives (BC). Denoting an annuity of this description by A^{BC}, and making the requisite substitutions in the last equation, we have

$$\rceil_{\substack{\mathrm{BC}\\ \mathrm{1}}}\mathrm{A}=\tfrac{1}{2}\left(\mathrm{A}-\mathrm{ABC}-\frac{{}^{1}b}{b}\cdot\mathrm{A}^{\mathrm{^1BC}}+\frac{{}^{1}c}{c}\mathrm{A}^{\mathrm{B^1C}}\right)\quad . \quad . \quad . \quad . \quad (3).$$

As this is one of the fundamental problems, upon which the solutions of the others depend, I will add two other modes of solving it.

The probability of the survivance of the joint lives (C) over the joint lives (B) taking place in the nth year is

$$\frac{{}^{n-1}b-{}^{n}b}{b}\cdot\frac{{}^{n-1}c+{}^{n}c}{2c},\ \text{or}\ \frac{{}^{n-1}b-{}^{n}b}{b}\cdot\frac{{}^{n-\frac{1}{2}}c}{c}.$$

But
$$\frac{{}^{n-1}b-{}^{n}b}{b}\cdot\frac{{}^{n-\frac{1}{2}}c}{c}=\frac{{}^{n-1}b^{n-\frac{1}{2}}c-{}^{n}b^{n-\frac{1}{2}}c}{bc}=\frac{{}^{n-1}(b^{\frac{1}{2}}c)-{}^{n-1}({}^{1}b^{\frac{1}{2}}c)}{bc}$$
$$=\frac{{}^{\frac{1}{2}}c}{c}\left(\frac{{}^{n-1}(b^{\frac{1}{2}}c)}{(b^{\frac{1}{2}}c)}-\frac{{}^{n-1}({}^{1}b^{\frac{1}{2}}c)}{({}^{1}b^{\frac{1}{2}}c)}\frac{{}^{1}b}{b}\right).$$

Multiply by $\rceil_{n-1}\mathrm{A}$, and we get

$$\rceil_{\substack{\mathrm{BC}\\ \mathrm{1}}}\mathrm{A}=\frac{{}^{\frac{1}{2}}c}{c}\left(S\cdot\frac{{}^{n-1}(b^{\frac{1}{2}}c)}{(b^{\frac{1}{2}}c)}\rceil_{n-1}\mathrm{A}-S\cdot\frac{{}^{n-1}({}^{1}b^{\frac{1}{2}}c)}{({}^{1}b^{\frac{1}{2}}c)}\rceil_{n-1}\mathrm{A}\cdot\frac{{}^{1}b}{b}\right)$$
$$=\frac{{}^{\frac{1}{2}}c}{c}\left(\mathrm{A}^{\mathrm{B^{\frac{1}{2}}C}}-\mathrm{A}^{\mathrm{^1B^{\frac{1}{2}}C}}\cdot\frac{{}^{1}b}{b}\right)\quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Again, as the first payment of the annuity $\rceil_{\substack{\mathrm{BC}\\ \mathrm{1}}}\mathrm{A}$ is to be made at the end of the year in which the survivance of the joint lives (C) over the joint lives (B) takes place, we may suppose it to fall due, upon an average, six months after that event. But an annuity has the same value, whether it be payable annually, half-yearly, or quarterly, or, indeed, at any other interval, if the next payment falls due at the expiration of one-half of the period intervening between the payments; and its value in that case may always be expressed by that of an annuity commencing immediately and payable momently.* Denoting, then, a momently annuity on the joint lives (A) by $\underset{\circ}{\mathrm{A}}$, it appears that

* It is, I believe, sometimes the practice, in making the periodical valuations of an Assurance Office, to substitute for *half-yearly* and *quarterly* premiums the corresponding *annual* payments. But as these half-yearly and quarterly policies, one with another, will become renewable at the expiration of one-half of the interval of renewal, it follows, from the circumstance above mentioned, that this substitution is not only unnecessary, but absolutely incorrect. The proper course is, to take the sum of the payments due in the year on each policy, whether one, two, or four, and value them as if they were payable altogether at the expiration of half a year. Of course, these remarks do not apply when the exact period at which the next premium becomes due, on each policy, is taken into account. In that case, the substitution referred to would be quite correct.

$$\rceil_{\mathrm{I}}^{\mathrm{BC}}\mathrm{A}=\rceil_{\mathrm{I}}^{\mathrm{BC}}\underset{\circ}{\mathrm{A}}=S\cdot\frac{{}^{n-1}b-{}^{n}b}{b}\cdot\frac{{}^{n-\frac{1}{2}}c}{c}\cdot\rceil^{n-\frac{1}{2}}\underset{\circ}{\mathrm{A}}.$$

That is, an annuity, the first payment of which is to be made at the end of the year in which the survivorship takes place, and to be paid annually thereafter, may be considered equivalent to a momently annuity commencing at the instant of survivorship.

Substituting in the last expression ${}^{n-\frac{1}{2}}(b_{\frac{1}{2}})$ for ${}^{n-1}b$, and ${}^{n-\frac{1}{2}}({}^{\frac{1}{2}}b)$ for ${}^{n}b$, we have

$$\rceil_{\mathrm{I}}^{\mathrm{BC}}\mathrm{A}=\frac{b_{\frac{1}{2}}}{b}S\frac{{}^{n-\frac{1}{2}}(b_{\frac{1}{2}}c)}{(b_{\frac{1}{2}}c)}\rceil^{n-\frac{1}{2}}\underset{\circ}{\mathrm{A}}-\frac{{}^{\frac{1}{2}}b}{b}S\frac{{}^{n-\frac{1}{2}}({}^{\frac{1}{2}}bc)}{({}^{\frac{1}{2}}bc)}\rceil^{n-\frac{1}{2}}\underset{\circ}{\mathrm{A}}.$$

The expression following the symbol of summation may be considered as a momently annuity on the joint lives (A), commencing with nothing, but increasing uniformly, by infinitely small increments at the rate of £1 per annum during the joint lives ($B_{\frac{1}{2}}C$) or (${}^{\frac{1}{2}}BC$). Denoting this by $\underset{\circ}{\mathrm{A}}^{\mathrm{B}_{\frac{1}{2}}\mathrm{C}}$ or $\underset{\circ}{\mathrm{A}}^{\frac{1}{2}\mathrm{BC}}$ we have

$$\rceil_{\mathrm{I}}^{\mathrm{BC}}\mathrm{A}=\frac{1}{b}\left(\underset{\circ}{\mathrm{A}}^{\mathrm{B}_{\frac{1}{2}}\mathrm{C}}\,b_{\frac{1}{2}}-\underset{\circ}{\mathrm{A}}^{\frac{1}{2}\mathrm{BC}}\,{}^{\frac{1}{2}}b\right)\quad.\quad.\quad.\quad.\quad.\quad(5).$$

PROBLEM II.

To determine ($\rceil_{\mathrm{II\,I}}^{\overline{\mathrm{BC}}}\mathrm{A}$) the present value of an annuity on the joint lives A_1, A_2, A_3, . . . after the failure of the joint lives B_1, B_2, B_3, . . . provided that event shall be preceded by the failure of the joint lives C_1, C_2, C_3, . . .

Solution.

$$\rceil_{\mathrm{I}}^{\mathrm{BC}}\mathrm{A}+\rceil_{\mathrm{II\,I}}^{\overline{\mathrm{BC}}}\mathrm{A}=\rceil_{\mathrm{B}}\mathrm{A}$$
$$=\mathrm{A}-\mathrm{AB}$$
$$\therefore\ \rceil_{\mathrm{II\,I}}^{\overline{\mathrm{BC}}}\mathrm{A}=\mathrm{A}-\mathrm{AB}-\rceil_{\mathrm{I}}^{\mathrm{BC}}\mathrm{A}.$$

PROBLEM III.

To determine ($\rceil_{\mathrm{II\,III\,I}}^{\mathrm{B\,C\,D}}\mathrm{A}$) the value of an annuity on the joint lives A_1, A_2, A_3, . . . after the failure of the joint lives B_1, B_2, B_3, . . . provided that event shall be preceded by the failure of the joint lives C_1, C_2, C_3, . . . during the existence of the joint lives D_1, D_2, D_3, . . .

Solution.

The contingent benefit stated in the problem is in effect a deferred annuity on the joint lives A, dependent upon the failure

of the joint lives C during the existence of the joint lives BD, and then further deferred until the failure of the joint lives B. If to this annuity there be added an annuity on the joint lives A, commencing immediately on the failure of the joint lives C during the existence of the joint lives BD, but ceasing on the failure of the joint lives B, the result will be an annuity on the joint lives A after the failure of the joint lives C during the existence of the joint lives BD. That is to say—

$$\rceil_{\substack{\mathrm{B}\;\mathrm{CD}\\ \mathrm{II.III}\;\mathrm{I}}}\mathrm{A}+\rceil_{\substack{\mathrm{CD}\\ \mathrm{I}}}\mathrm{AB}=\rceil_{\substack{\mathrm{CBD}\\ \mathrm{I}}}\mathrm{A}$$

$$\therefore \rceil_{\substack{\mathrm{B}\;\mathrm{CD}\\ \mathrm{II.III}\;\mathrm{I}}}\mathrm{A}=\rceil_{\substack{\mathrm{CBD}\\ \mathrm{I}}}\mathrm{A}-\rceil_{\substack{\mathrm{CD}\\ \mathrm{I}}}\mathrm{AB}.$$

The two terms of the second member of the last equation are resolved by Problem I.

Problem IV.

To determine $(\rceil_{\substack{\overline{\mathrm{B\,C\,D}}\\ \mathrm{III\,II\,I}}}\mathrm{A})$ the value of an annuity on the joint lives A_1, A_2, A_3, . . . after the failure of the joint lives B_1, B_2, B_3, . . . provided that event shall be preceded by the failure of the joint lives C_1, C_2, C_3, . . . after the failure of the joint lives D_1, D_2, D_3, . . .

Solution.

$$\rceil_{\substack{\overline{\mathrm{B\,C\,D}}\\ \mathrm{III\,II\,I}}}\mathrm{A}+\rceil_{\substack{\mathrm{B}\;\mathrm{C}\;\mathrm{D}\\ \mathrm{II.III}\;\mathrm{I}}}\mathrm{A}=\rceil_{\substack{\overline{\mathrm{BC}}\\ \mathrm{II\,I}}}\mathrm{A}=\mathrm{A}-\mathrm{AB}-\rceil_{\substack{\mathrm{BC}\\ \mathrm{I}}}\mathrm{A}\ \text{(Prob. II.)}$$

$$\therefore \rceil_{\substack{\overline{\mathrm{B\,C\,D}}\\ \mathrm{III\,II\,I}}}\mathrm{A}=\mathrm{A}-\mathrm{AB}-\rceil_{\substack{\mathrm{BC}\\ \mathrm{I}}}\mathrm{A}-\rceil_{\substack{\mathrm{B}\;\mathrm{C}\;\mathrm{D}\\ \mathrm{II.III}\;\mathrm{I}}}\mathrm{A}$$

$$=\mathrm{A}-\mathrm{AB}-\rceil_{\substack{\mathrm{BC}\\ \mathrm{I}}}\mathrm{A}-\rceil_{\substack{\mathrm{CBD}\\ \mathrm{I}}}\mathrm{A}+\rceil_{\substack{\mathrm{CD}\\ \mathrm{I}}}\mathrm{AB}.$$

Problem V.

To determine $(\underset{1}{\mathfrak{A}}\mathbb{B})$ the present value of £1 payable upon the ailure of the joint lives A_1, A_2, A_3, . . . during the existence of the joint lives B_1, B_2, B_3, . . .

Solution.

Adopting the approximative formula used in the solution of Problem I. for the chance of the joint lives (A) failing before the joint lives (B) during the nth year, we shall have

$$\underset{1}{\mathfrak{A}}\mathbb{B}=S\cdot\frac{{}^{n-1}a-{}^{n}a}{a}\cdot\frac{{}^{n-1}b+{}^{n}b}{2b}\cdot v^{n}$$

$$=S\cdot\frac{{}^{n-1}(ab)-{}^{n}(ab)+{}^{n-1}a\,{}^{n}b-{}^{n}a\,{}^{n-1}b}{2(ab)}\cdot v^{n}$$

$$=\tfrac{1}{2}S\left(\frac{{}^{n-1}(ab)-{}^{n}(ab)}{(ab)}\cdot v^{n}+\frac{{}^{n}(a_1b)}{(a_1b)}\cdot v^{n}\cdot\frac{a_1}{a}-\frac{{}^{n}(ab_1)}{(ab_1)}\cdot v^{n}\cdot\frac{b_1}{b}\right)$$

$$=\tfrac{1}{2}\left(\mathfrak{A}\mathfrak{B}+\frac{a_1}{a}\cdot \mathrm{A}_1\mathrm{B}-\frac{b_1}{b}\mathrm{AB}_1\right)\quad\ldots\ldots\quad(1).$$

When there is one life only in each set, this formula is exact, upon the hypothesis of a uniform distribution of the deaths of each year, and is identical with the formula of Mr. Milne. In other cases, however, it will be found sufficiently accurate for all practical purposes.

Again, $({}^{n-1}a-{}^{n}a).{}^{n-\frac{1}{2}}b.v^{n}={}^{n}(a_1b_{\frac{1}{2}})v^{n}-{}^{n}(ab_{\frac{1}{2}})v^{n}$, whence

$$\underset{1}{\mathfrak{A}}\mathbb{B}=S\frac{{}^{n}(a_1b_{\frac{1}{2}})v^{n}}{(ab)}-S\frac{{}^{n}(ab_{\frac{1}{2}})v^{n}}{(ab)}$$

$$=\left(\mathrm{A}_1\mathrm{B}_{\frac{1}{2}}\cdot\frac{a_1}{a}-\mathrm{AB}_{\frac{1}{2}}\right)\frac{b_{\frac{1}{2}}}{b}\quad\ldots\ldots\quad(2).*$$

Lastly, $({}^{n-1}a-{}^{n}a)^{n-\frac{1}{2}}b.v^{n}=\{{}^{n-\frac{1}{2}}(a_{\frac{1}{2}}b).v^{n-\frac{1}{2}}-{}^{n-\frac{1}{2}}({}^{\frac{1}{2}}ab)v^{n-\frac{1}{2}}\}v^{\frac{1}{2}}$

$$\underset{1}{\mathfrak{A}}\mathbb{B}=\frac{v^{\frac{1}{2}}}{a}\left(S\frac{{}^{n-\frac{1}{2}}(a_{\frac{1}{2}}b)v^{n-\frac{1}{2}}}{(a_{\frac{1}{2}}b)}a_{\frac{1}{2}}-S\frac{{}^{n-\frac{1}{2}}({}^{\frac{1}{2}}ab)v^{n-\frac{1}{2}}}{({}^{\frac{1}{2}}ab)}\cdot{}^{\frac{1}{2}}a\right)$$

$$=\frac{v^{\frac{1}{2}}}{a}\left(\underbrace{\mathrm{A}_{\frac{1}{2}}\mathrm{B}}_{\circ}.\,a_{\frac{1}{2}}-{}^{\frac{1}{2}}\underbrace{\mathrm{AB}}_{\circ}.{}^{\frac{1}{2}}a\right)\quad\ldots\ldots\ldots\quad(3),$$

the expression $\underbrace{\mathrm{AB}}_{\circ}$ denoting the value of an annuity payable momently on the joint lives (AB). If we omit the factor $v^{\frac{1}{2}}$, this formula will give the value of the assurance payable at the instant of death.

PROBLEM VI.

To determine $(\underset{\text{II}}{\mathfrak{A}}\mathbb{B})$ the present value of £1 payable upon the failure of the joint lives $A_1, A_2, A_3, \ldots$ provided the joint lives $B_1, B_2, B_3, \ldots$ shall have failed previously.

Solution.

$$\underset{\text{I}}{\mathfrak{A}}\mathbb{B}+\underset{\text{II}}{\mathfrak{A}}\mathbb{B}=\mathfrak{A}$$

$$\therefore\ \underset{\text{II}}{\mathfrak{A}}\mathbb{B}=\mathfrak{A}-\underset{\text{I}}{\mathfrak{A}}\mathbb{B}.$$

* This formula is given by Mr. Gompertz, in his *Analysis Applicable to the Estimation of Life Contingencies.*

Problem VII.

To determine $(\underset{\mathrm{II.III}}{\mathfrak{A}}\mathbb{B}\underset{\mathrm{I}}{\mathbb{C}})$ the present value of £1 payable on the failure of the joint lives $A_1, A_2, A_3, \ldots$ provided the joint lives $C_1, C_2, C_3, \ldots$ shall have failed previously during the existence of the joint lives $B_1, B_2, B_3, \ldots$

Solution.

$$\urcorner_{\underset{\mathrm{I}}{BC}}A+\underset{\mathrm{II.III}}{\mathfrak{A}}\mathbb{B}\underset{\mathrm{I}}{\mathbb{C}}\left(1+\frac{1}{r}\right)=\underset{\mathrm{I}}{\mathfrak{C}}\mathbb{A}\mathbb{B}\left(1+\frac{1}{r}\right)$$

$$\underset{\mathrm{II.III}}{\mathfrak{A}}\mathbb{B}\underset{\mathrm{I}}{\mathbb{C}}=\underset{\mathrm{I}}{\mathfrak{C}}\mathbb{A}\mathbb{B}-\urcorner_{\underset{\mathrm{I}}{BC}}A(1-v).$$

Problem VIII.

To determine $(\underset{\mathrm{III}}{\mathfrak{A}}\mathbb{B}\underset{\mathrm{II}}{\mathbb{C}})$ the present value of £1 payable on the failure of the joint lives $A_1, A_2, A_3, \ldots$ provided the joint lives $C_1, C_2, C_3, \ldots$ shall have failed previously after the failure of the joint lives $B_1, B_2, B_3, \ldots$

Solution.

$$\underset{\mathrm{III}}{\mathfrak{A}}\mathbb{B}\underset{\mathrm{II}}{\mathbb{C}}+\underset{\mathrm{II.III}}{\mathfrak{A}}\mathbb{B}\underset{\mathrm{I}}{\mathbb{C}}=\underset{\mathrm{II}}{\mathfrak{A}}\mathbb{C}=\mathfrak{A}-\underset{\mathrm{I}}{\mathfrak{A}}\mathbb{C}$$

$$\underset{\mathrm{III}}{\mathfrak{A}}\mathbb{B}\underset{\mathrm{II}}{\mathbb{C}}=\mathfrak{A}-\underset{\mathrm{I}}{\mathfrak{A}}\mathbb{C}-\underset{\mathrm{II.III}}{\mathfrak{A}}\mathbb{B}\underset{\mathrm{I}}{\mathbb{C}}$$

$$=\mathfrak{A}-\underset{\mathrm{I}}{\mathfrak{A}}\mathbb{C}-\underset{\mathrm{I}}{\mathfrak{C}}\mathbb{A}\mathbb{B}+\urcorner_{\underset{\mathrm{I}}{BC}}A(1-v).$$

Problem IX.

To determine $(\underset{\mathrm{II.III}}{\mathfrak{A}}\mathbb{B}\underset{\underset{\mathrm{II}}{c}}{\mathbb{C}}\underset{\mathrm{I}}{\mathbb{D}})$ the present value of £1 payable on the failure of the joint lives $A_1, A_2, A_3, \ldots$ provided that event shall occur after the failure of the joint lives $D_1, D_2, D_3, \ldots$ and before the failure of the joint lives $B_1, B_2, B_3 \ldots$; and provided the joint lives $C_1, C_2, C_3, \ldots$ shall survive the joint lives $D_1, D_2, D_3, \ldots$

Solution.

The contingent benefit stated in this problem, is, in effect, an assurance on the joint lives $A_1, A_2, A_3, \ldots$ against the joint lives $B_1, B_2, B_3 \ldots$ deferred until the failure of the joint lives $D_1, D_2, D_3, \ldots$ during the existence of the joint lives $C_1, C_2, C_3 \ldots$ The value of the contingency due to the nth year will therefore be $\frac{{}^{n-1}d-{}^{n}d}{d}\cdot\frac{{}^{n-1}c+{}^{n}c}{2c}\urcorner_{n-\frac{1}{2}}\underset{\mathrm{I}}{\mathfrak{A}}\mathbb{B}$, on the assumption that the survivance of the joint lives (C) over the joint lives (D), if occurring

in the nth year, will take place in the middle of that year. We have, therefore,

$$\underset{\overset{\text{c}}{\text{n}}}{\overset{}{\mathfrak{A}}}_{\text{nm}}\mathfrak{B}\mathfrak{C}\underset{1}{\mathfrak{D}}=S\cdot\frac{{}^{n-1}d-{}^{n}d}{d}\cdot\frac{{}^{n-1}c+{}^{n}c}{2c}\urcorner_{n-\frac{1}{2}}\underset{1}{\mathfrak{A}}\mathfrak{B}\quad . \quad . \quad . \quad . \quad (1)$$

Now, referring to the third solution of Problem V., we find that

$$\underset{1}{\mathfrak{A}}\mathfrak{B}=\frac{v^{\frac{1}{2}}}{a}\left(S\cdot\frac{{}^{n-\frac{1}{2}}(a_{\frac{1}{2}}b)v^{n-\frac{1}{2}}}{a_{\frac{1}{2}}b}\cdot a_{\frac{1}{2}}-S\cdot\frac{{}^{n-\frac{1}{2}}({}^{\frac{1}{2}}ab)v^{n-\frac{1}{2}}}{{}^{\frac{1}{2}}ab}\cdot{}^{\frac{1}{2}}a\right).$$

But as the assurance is to be deferred $n-\frac{1}{2}$ years, the summation, instead of commencing with $n=1$, will begin with $n-\frac{1}{2}+1$, or $n+\frac{1}{2}$; that is, the first terms of the variable quantities in the last expression will be

$$\frac{{}^{n}(a_{\frac{1}{2}}b)v^{n}}{a_{\frac{1}{2}}b}\text{ and }\frac{{}^{n}({}^{\frac{1}{2}}ab)v^{n}}{{}^{\frac{1}{2}}ab}.$$

The result will evidently be in each case an annuity deferred $n-1$ years.

We have, therefore,

$$\urcorner_{n-\frac{1}{2}}\underset{1}{\mathfrak{A}}\mathfrak{B}=\frac{v^{\frac{1}{2}}}{a}\left(\urcorner_{n-1}\text{A}_{\frac{1}{2}}\text{B}\,.\,a_{\frac{1}{2}}-\urcorner_{n-1}{}^{\frac{1}{2}}\text{AB}.{}^{\frac{1}{2}}a\right),$$

and substituting this value in (1), it appears that

$$\underset{\overset{\text{c}}{\text{n}}}{\overset{}{\mathfrak{A}}}_{\text{nm}}\mathfrak{B}\mathfrak{C}\underset{1}{\mathfrak{D}}=\frac{v^{\frac{1}{2}}}{a}\left(S\frac{{}^{n-1}d-{}^{n}d}{d}\cdot\frac{{}^{n-1}c+{}^{n}c}{2c}\urcorner_{n-1}\text{A}_{\frac{1}{2}}\text{B}\,.\,a_{\frac{1}{2}}-S\cdot\frac{{}^{n-1}d-{}^{n}d}{d}\cdot\frac{{}^{n-1}c+{}^{n}c}{2c}\urcorner_{n-1}{}^{\frac{1}{2}}\text{AB}.{}^{\frac{1}{2}}a\right)$$

$$=\frac{v^{\frac{1}{2}}}{a}\left(\urcorner_{\underset{1}{\text{CD}}}\text{A}_{\frac{1}{2}}\text{B}\,.\,a_{\frac{1}{2}}-\urcorner_{\underset{1}{\text{CD}}}{}^{\frac{1}{2}}\text{AB}.{}^{\frac{1}{2}}a\right)\quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

PROBLEM X.

To determine ($\underset{\text{m}}{\mathfrak{A}}\mathfrak{B}\underset{1}{\mathfrak{C}}\underset{\text{n}}{\mathfrak{D}}$) the present value of £1 payable on the failure of the joint lives A_1, A_2, A_3, . . . provided that event shall occur after the failure of the joint lives D_1, D_2, D_3, . . . and before the failure of the joint lives B_1, B_2, B_3 . . .; and provided the joint lives C_1, C_2, C_3, . . . shall fail before the joint lives D_1, D_2, D_3, . . .

Solution.

$$\underset{\overset{\text{c}}{\text{n}}}{\overset{}{\mathfrak{A}}}_{\text{nm}}\mathfrak{B}\mathfrak{C}\underset{1}{\mathfrak{D}}+\underset{\text{m}}{\mathfrak{A}}\mathfrak{B}\underset{1}{\mathfrak{C}}\underset{\text{n}}{\mathfrak{D}}=\underset{\text{n}}{\mathfrak{A}}\mathfrak{B}\underset{1}{\mathfrak{D}}=\underset{1}{\mathfrak{A}}\mathfrak{B}-\underset{1}{\mathfrak{A}}\mathfrak{B}\mathfrak{D}\text{ (Milne, XIX.)}$$

$$\therefore\ \underset{\text{m}}{\mathfrak{A}}\mathfrak{B}\underset{1}{\mathfrak{C}}\underset{\text{n}}{\mathfrak{D}}=\underset{1}{\mathfrak{A}}\mathfrak{B}-\underset{1}{\mathfrak{A}}\mathfrak{B}\mathfrak{D}-\underset{\overset{\text{c}}{\text{n}}}{\overset{}{\mathfrak{A}}}_{\text{nm}}\mathfrak{B}\mathfrak{C}\underset{1}{\mathfrak{D}}.$$

PROBLEM XI.

To determine $(\underset{\text{iii.iv}}{\mathfrak{A}}\underset{\text{ii.iii}}{\mathbb{B}}\underset{\text{i}}{\mathbb{C}}\mathbb{D})$ the present value of £1 payable on the failure of the joint lives A_1, A_2, A_3, . . . provided that event shall occur after the failure of the joint lives B_1, B_2, B_3 . . .; and provided the joint lives C_1, C_2, C_3, . . . shall fail before the joint lives B_1, B_2, B_3, . . . and before the joint lives D_1, D_2, D_3, . . .

Solution.

$$\urcorner_{\underset{\text{ii.iii}}{B}\underset{\text{i}}{C}D}A+\underset{\text{iii.iv}}{\mathfrak{A}}\underset{\text{ii.iii}}{\mathbb{B}}\underset{\text{i}}{\mathbb{C}}\mathbb{D}\left(1+\frac{1}{r}\right)=\underset{\substack{\text{ii.iii}\\ d\\ \text{ii}}}{\mathfrak{B}}\mathbb{A}\underset{\text{i}}{\mathbb{C}}\mathbb{D}\left(1+\frac{1}{r}\right)$$

$$\therefore\ \underset{\text{iii.iv}}{\mathfrak{A}}\underset{\text{ii.iii}}{\mathbb{B}}\underset{\text{i}}{\mathbb{C}}\mathbb{D}=\underset{\substack{\text{ii.iii}\\ d\\ \text{ii}}}{\mathfrak{B}}\mathbb{A}\underset{\text{i}}{\mathbb{C}}\mathbb{D}-\urcorner_{\underset{\text{ii.iii}}{B}\underset{\text{i}}{C}D}A(1-v)$$

$$=\underset{\substack{\text{ii.iii}\\ d\\ \text{ii}}}{\mathfrak{B}}\mathbb{A}\underset{\text{i}}{\mathbb{C}}\mathbb{D}-(\urcorner_{\underset{\text{i}}{C}BD}A-\urcorner_{\underset{\text{i}}{C}D}AB)(1-v).$$

PROBLEM XII.

To determine $(\underset{\text{iv}}{\mathfrak{A}}\underset{\text{iii}}{\mathbb{B}}\underset{\text{ii}}{\mathbb{C}}\underset{\text{i}}{\mathbb{D}})$ the present value of £1 payable on the joint lives A_1, A_2, A_3, . . . failing after the failure of the joint lives B_1, B_2, B_3, . . . provided that the joint lives C_1, C_2, C_3, . . . shall fail before the joint lives B_1, B_2, B_3, . . . and after the joint lives D_1, D_2, D_3, . . .

Solution.

$$\underset{\text{iv}}{\mathfrak{A}}\underset{\text{iii}}{\mathbb{B}}\underset{\text{ii}}{\mathbb{C}}\underset{\text{i}}{\mathbb{D}}+\underset{\text{iii.iv}}{\mathfrak{A}}\underset{\text{ii.iii}}{\mathbb{B}}\underset{\text{i}}{\mathbb{C}}\mathbb{D}=\underset{\text{iii}}{\mathfrak{A}}\underset{\text{ii}}{\mathbb{B}}\underset{\text{i}}{\mathbb{C}}=\mathfrak{A}-\underset{\text{i}}{\mathfrak{A}}\mathbb{B}-\underset{\text{i}}{\mathfrak{B}}\mathbb{A}\mathbb{C}+\urcorner_{\underset{\text{i}}{B}C}A(1-v)$$

$$\underset{\text{iv}}{\mathfrak{A}}\underset{\text{iii}}{\mathbb{B}}\underset{\text{ii}}{\mathbb{C}}\underset{\text{i}}{\mathbb{D}}=\mathfrak{A}-\underset{\text{i}}{\mathfrak{A}}\mathbb{B}-\underset{\text{i}}{\mathfrak{B}}\mathbb{A}\mathbb{C}+\urcorner_{\underset{\text{i}}{B}C}A(1-v)-\underset{\text{iii.iv}}{\mathfrak{A}}\underset{\text{ii.iii}}{\mathbb{B}}\underset{\text{i}}{\mathbb{C}}\mathbb{D}$$

$$=\mathfrak{A}-\underset{\text{i}}{\mathfrak{A}}\mathbb{B}-\underset{\text{i}}{\mathfrak{B}}\mathbb{A}\mathbb{C}-\underset{\substack{\text{ii.iii}\\ d\\ \text{ii}}}{\mathfrak{B}}\mathbb{A}\underset{\text{i}}{\mathbb{C}}\mathbb{D}+(\urcorner_{\underset{\text{i}}{B}C}A+\urcorner_{\underset{\text{i}}{C}BD}A-\urcorner_{\underset{\text{i}}{C}D}AB)(1-v).$$

On a Table for the Formation of Logarithms and Anti-Logarithms to Twelve Places. By PETER GRAY, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

[Read before the Institute, 27th December, 1864.]

WITH the exception of the publication, on a more limited scale (and which will be hereafter referred to), of that which is now to be developed, there exists at present no ready and practical method of forming to more than seven places the logarithms of numbers of more than seven or eight figures. The great extent of the tables requisite, if formed on the plan of our present seven-figure tables, is not only likely ever to prove a bar to their construction, but it

would also render them too cumbrous, if constructed, to be easily and readily used. In the present method, the end in view is sought to be attained in another way. The principle of the method is the resolution of the number whose logarithm is required, by a direct and easy process, into factors of a peculiar form, the logarithms of which, to the requisite extent, admit of easy tabulation. The logarithms of the factors, then, being taken from the table, their sum is the logarithm of the given number.

It may be said, that the foregoing is the principle employed in other methods that have been proposed for the same purpose. This is quite true. And I think I may venture to say that it is the principle that must of necessity be employed in every method that may hereafter be proposed. What constitutes the superiority claimed for the present over previous methods, is the peculiar form of the factors into which the number is resolved, and the resulting simplicity of the process by which the resolution is effected. I reserve what I have to say on the history of the method till the table has been described, and the manner of using it exemplified.

Description and use of the Table.

The table, which is adapted for the formation of logarithms and anti-logarithms of twelve places, consists of three columns, headed I., II., III., respectively. We require a Col. IV.; but it is unnecessary to insert it, as, if separately exhibited, it would consist of merely the first three figures (made true in the last place) of the corresponding values in Col. III. The argument, which occupies the small side column, headed n, extends from 000 to 999. The values in the several columns are related to the corresponding argument value as follows:—If n denote any value in the argument column, then we have corresponding,

in Col. I., $\log(1+{\cdot}001n)$;
„ II., „ $(1+{\cdot}001^2n)$;
„ III., „ $(1+{\cdot}001^3n)$;
„ IV., „ $(1+{\cdot}001^4n)$.

For example, corresponding to 576 we have,

in Col. I., log 1·576;
„ II., „ 1·000,576;
„ III., „ 1·000,000,576;
„ IV., „ 1·000,000,000,576.

The ciphers following the decimal point, and preceding the figures exhibited in Cols. II. and III., are omitted, so that the last figure in each tabular value occupies the twelfth decimal place.

The Auxiliary Table contains the logarithms and the cologarithms (that is, the logarithms of the reciprocals) of the natural numbers 2 to 9.

Problem I.

To find the logarithm of any given number, to twelve places.

First. When the first figure of the given number is unity.

1. Insert the decimal point after the first figure, and make twelve decimal places, either by annexing ciphers or by cutting off figures that extend beyond the twelfth place, as the case may require. It is convenient also to separate the decimal into periods of three figures, either in the usual way, by points or commas, or, if paper ruled in squares be used, by deepening every third vertical line. The number thus modified I call the prepared number.

2. Cut off the leading figure and the first period of the prepared number for a divisor, and let the remaining portion of the number be the dividend. Divide now till three quotient figures have been obtained; observing that, as the first quotient figure must be got from the first four figures of the dividend, if the divisor will not go in those four figures, the first quotient figure will be a cipher. This division will use up the second and third periods of the prepared number.

3. Annex to the remainder of the previous division the remaining period of the prepared number, for a new dividend; and for a new divisor, annex to the first the second period of the prepared number (subject to an occasional small correction in its last figure, as will be presently explained). Set down the divisor thus formed in juxtaposition with its dividend, and continue the division, now necessarily in the contracted form, the first step being the pointing off of the last figure of the divisor. Six quotient figures will be obtained, which are to be arranged in triads.

4. The first period of the prepared number (that is, the decimal portion of the first divisor) and the three quotients thus form four triads, with which the four columns of the principal table are to be entered in order, and the results taken out. The sum of these will be the mantissa of the required logarithm, to which the characteristic, to be determined by reference to the given number, must be prefixed.

The correction above referred to, as occasionally due to the second divisor, arises as follows:—The true and complete second divisor, if no contraction were used, would be the number consisting of the leading unit and the first three periods of the prepared

number, diminished by the remainder of the first division; and it is this diminished number, when curtailed of its last period, and made true in the last place retained, which ought to form our second divisor. It is necessary, therefore, when the utmost attainable accuracy is desired, to attend to the effect of the diminution referred to on the portion of the divisor that we retain. It will soon be perceived that this effect must be in all cases very small. In point of fact, the remainder—the number by which the complete divisor has to be diminished—consisting generally of only three figures, and these occupying the same place in the decimal scale as the period which is cut off for the formation of our contracted divisor, will frequently have an appreciable effect on that period alone, so as to leave the last figure of the adjoining period unaltered, or even not to interfere with the increase which is due to it in consequence of the first figure of the period cut off being 5 or more. Never, in any case, is the correction greater than the diminution by a unit of the last figure of the second period. And when it is recollected that this last figure is cut off at the first step, and so contributes only the carriage from it to the subsequent operation, it is obvious that the correction, even in cases where it is found to be due, will only occasionally exercise any influence on the final result. It is nevertheless so easy to ascertain, in any particular case, whether or not the correction is due, that I do not feel justified in suggesting that it should be overlooked.

Secondly. When the first figure of the given number is other than unity.

1. Multiply or divide the given number by any number consisting of a single digit that will give a result having unity for its first figure; and proceed with this result, as regards the resolving process, as above directed. The number thus employed I call the preparing number.

2. Take from the Auxiliary Table, according as the preparing number was used as a divisor or a multiplier, its logarithm or its co-logarithm, and include it in the addition with the values taken from the principal table. The sum, with the proper characteristic prefixed, will, as before, be the logarithm required.

It is obvious that any number whatsoever may as above be brought within the compass of our table: division by its first digit will bring it into the required form. But we are by no means restricted to a single mode of preparation. Every number admits of no fewer than four or five, and two of them may be employed when verification is required.

I now give some examples :—

Ex. 1.—Required the logarithm of 134·514,347,22.

1·345)143 472 200	106
134 5	
8 972	
8 070	
1·345 14~~3~~)902 200	670
807 086	
95 114	
94 160	
954	709
942	
12	
128 722 284 338	345 Col. I.
46 032 775	106 „ II.
290 977	670 „ III.
308	709 „ IV.
2·128 768 608 398	=log. req.

13 451 434 722	÷9
1·494)603 858 000	404
597 6	
6 258	
5 976	
1·494 60~~4~~)282 000	188
149 460	
132 540	
119 568	
12 972	
11 957	
1 015	679
897	
118	
105	
13	
954 242 509 439	log 9
174 350 597 479	494 Col. I.
175 419 538	404 „ II.
81 647	188 „ III.
295	679 „ IV.
2·128 768 608 398	=log. req.

I give two solutions of this example, which verify each other. The operations are so simple, they admit of such lucid arrangement, and the steps are consequently so easily followed, that special description, after the full explanations that have been given, seems to be unnecessary. I have used division by 9 in the second solution; but division by 7 or 8, or multiplication by 8 or 9, would have answered equally well. One or more of these may be tried, as exercises.

In the first solution, the last figure of the second period of the prepared number is unchanged in the second divisor. The subtraction of the remainder, 902, from 3,472, gives 2,570; and cutting off the last period, the 2 becomes 3. In the second solution, the subtraction of 282 from 3,858 gives 3,576; and on cutting off the last period, the 3 becomes 4.

The results of the two foregoing operations agree in the last figure, but the agreement is fortuitous, for both are wrong. The last figure is truly 9. It is obvious that in such operations, where contracted division is used, and the logarithms employed are only approximately true, we must be prepared for some degree of uncertainty as to the last place. I am able to say, however, that, in the course of a pretty extensive use of these tables, I have to my knowledge met with only a single instance in which the error was so great as 2 in the last place. An error of 1, in either excess or defect, I admit, is far from rare; but I believe that, in a majority of cases, the logarithms formed are true to the nearest figure.*

Ex. 2.—Given $\pi = 3{\cdot}141{,}592{,}653{,}590$; required its logarithm.

3 141 592 653 590	÷3
1·047)197 551 197	188
104 7	
92 85	
83 76	
9 091	
8 376	
1·047 19$\cancel{7}$)715 197	682
628 318	
86 879	
83 776	
3 103	
2 094	
1 009	964
942	
67	
63	
4	

314 159 265 359 0	× 4
1·256)637 061 436	507
628 0	
9 061	
8 792	
1·256 63$\cancel{7}$)269 436	214
251 327	
18 109	
12 566	
5 543	
5 027	
516	410
503	
13	

477 121 254 720	log 3
19 946 681 679	047
81 639 689	188
296 189	682
419	964
0·497 149 872 696	$=\log \pi$

397 940 008 672	colog 4
98 989 639 401	256
220 131 504	507
92 939	214
178	410
0·497 149 872 694	$=\log \pi$

* Certainly more than this cannot be said of the results of the common seven-figure tables, when interpolation is used.

Two solutions are given, using division by 3, and multiplication by 4. Division by 2 and multiplication by 6 would give two more. The results differ by 2 in the twelfth place, the first being wrong and the second right. In fact, this is the instance referred to above, as being the only one in which I recollect to have met with an error so great as 2. And it arises in this case I find, chiefly, from the circumstance that the whole of the five logarithms, whose sum gives log. π, have been increased in their last figure. The concurrence of so many as five logarithms similarly affected must necessarily be extremely rare.

Ex. 3.—Given ε (the Naperian base) $=2{\cdot}718,281,828,459$; required log ε.

2 718 281 828 459	÷2	271 828 182 845 9	×4
1·359)140 914 230	103	1·087)312 731 384	287
135 9		217 4	
5 014		95 33	
4 077		86 96	
1·359 14~~0~~)937 230	689	8 371	
815 484		7 609	
121 746		1·087 31~~2~~)762 384	701
108 731		761 118	
13 015		1 266	
12 232		1 087	
783	576	179	165
680		109	
103		70	
95		65	
8		5	
301 029 995 664	log 2	397 940 008 672	colog 4
133 219 456 732	359	36 229 544 086	087
44 730 028	103	124 624 634	287
299 229	689	304 440	701
250	576	72	165
0·434 294 481 903	=M	0·434 294 481 904	=M

Two solutions are again given, and two more may be had by using 6 and 7 as multipliers. The result of the first is true in the

last figure; that of the second errs by a unit in the last place in excess. The logarithm formed is the modulus of the common system, and, as such, is usually denoted by M.

Ex. 4.—Required the logarithm of 54·3839.

543 839	×2	5 438 39	÷4
1·087)678 000 000	623	1·359)597 500 000	439
652 2		543 6	
25 80		53 90	
21 74		40 77	
4 060		13 130	
3 261		12 231	
1·087 67$\cancel{7}$)799 000	734	1·359 59$\cancel{7}$)899 000	661
761 374		815 758	
37 626		83 242	
32 630		81 576	
4 996		1 666	
4 351		1 360	
645	593	306	225
544		272	
101		34	
98		27	
3		7	
698 970 004 336	colog 2	602 059 991 328	log 4
36 229 544 086	087	133 219 456 732	359
270 481 216	623	190 613 441	439
318 772	734	287 069	661
258	593	98	225
1·735 470 348 668	log. req.	1·735 470 348 668	

I have selected this example because it is the one employed by Borda and his editor, Delambre, for the illustration of their methods. (See Borda's Tables, author's and editor's prefaces, pp. 16, 17, 77 to 81.) And I invite comparison of the above easy and direct processes, in which not a figure is suppressed, with those made use of by the distinguished mathematicians just named. Both the solutions here given are true in the last figure.

Ex. 5.—Required the logarithm of 67·383,036,620,64.

6 738 303 662 064	÷6	3 965 595 716 040	÷3
1·123)050 610 344	045	1·321)865 238 680	654
44 92		792 6	
5 690		72 63	
5 615		66 05	
1·123 05~~1~~)075 344	067	6 588	
67 383		5 284	
7 961		1·321 86~~4~~)1 304 680	987
7 861		1 189 678	
100	089	115 002	
90		105 749	
10		9 253	
		9 253	
778 151 250 384	log 6		
50 379 756 261	123	477 121 254 720	log 3
19 542 812	045	120 902 817 615	321
29 098	067	283 935 754	654
39	089	428 648	987
1·828 550 578 594	log. req.	2·598 308 436 737	log. req.

This example has been chosen to illustrate the case of the occurrence of ciphers in the first place of the several quotients.

Ex. 6.—Find the logarithm of 396·559,571,604.

This example affords an instance of what sometimes happens, namely, the encroachment by the remainder of the first division on the space occupied in previous examples by the second divisor. This occasions little inconvenience. The remedy is, to move the second divisor a little to the left.

We may notice here also, as confirmatory of a previous remark, that notwithstanding the comparative magnitude of the remainder in question, the effect of it in the formation of the second divisor is still no more than the abatement of a unit in its last figure.

There is another specialty in this example: the dividend is exhausted when the second triad of quotient figures has been obtained. There is, consequently, no entry in Col. IV.

Ex. 7.—Required the logarithm of 1·000,693,387,464.

1·000 69~~3~~)387 464	387
300 208	
87 256	
80 055	
7 201	
7 005	
196	196
000 300 861 839	693 Col. II.
168 072	387 „ III.
85	196 „ IV.
0·000 301 029 996	=log.req.

10 006 933 874 64	÷7
1·429)561 982 091	393
428 7	
133 28	
128 61	
4 672	
4 287	
1·429 56~~7~~)385 091	269
285 912	
99 179	
85 774	
13 405	
12 866	
539	377
429	
110	
100	
10	
845 098 040 014	log. 7
155 032 228 791	429
170 644 202	393
116 825	269
164	377
0·000 301 029 996	=log.req.

We might here proceed strictly according to the rule, cutting off the first period, 1·000, for the divisor. Obviously the effect would be, that the first quotient would be 693, and we should get for the second divisor the first two periods unaltered. We avoid needless work therefore by commencing with this divisor. In the second part of the process, still working by rule, we should take 000 as the argument for Col. I. But the result answering to this being 0, we simply pass over Col. I., and commence our entries with Col. II.

A second solution is given for verification, requiring many more figures than the first. Both are true in the last place. (See *Ex.* 14.)

When nine figures only are required in the logarithm, the process undergoes material simplification. The first divisor is used

throughout the resolving process, contraction, by its curtailment, commencing immediately after the formation of the first quotient; and there is no entry in Col. IV.

Ex. 8.—Required log π to nine places.

3 141 592 654	÷3
1·047)197 551	188
104 7	
92 85	
83 76	
9 091	
8 376	
715	683
628	
87	
84	
3	
477 121 255	log 3
19 946 682	047 Col. I.
81 640	188 „ II.
297	683 „ III.
0·497 149 874	=log π

314 159 265 4	×4
1·256)637 062	507
628 0	
9 062	
8 792	
270	215
251	
19	
13	
6	
397 940 009	colog 4
98 989 639	256
220 132	507
93	215
0·497 149 873	=log π

Two solutions are given, the second being true in the last figure. (See *Ex.* 2.)

Ex. 9.—Required the logarithm of ·123,456,79 to nine places.

1·234)567 900	460
493 6	
74 30	
74 04	
260	211
247	
13	
12	
1	
091 315 160	234 Col. I.
199 730	460 „ II.
092	211 „ III.
$\bar{1}$·091 514 982	=log. req.

123 456 79	×9
1·111)111 110	100
111 1	
10	009
045 757 491	colog 9
45 714 059	111
43 427	100
4	009
$\bar{1}$·091 514 981	

Of the two solutions here given, differing by one in the last place, I do not know which is the more correct. It so happens that the second is simplified by the use of a preparing number.

If six places in the mantissa of the logarithm suffice, the process is still further simplified. Contraction of the divisor here commences previous to the formation of the first quotient, and there is no argument for either Col. III. or Col. IV.

Ex. 10.—Required log. π to six places.

3 141 593	÷3	314 159 3×4	
1·047)198	189	1·25$\cancel{6}$ 637	507
105		628	
93		9	
84			
9			
477 121	log 3	397 940	colog 4
19 947	047 Col. I.	98 990	256 Col. I.
82	189 ,, II.	220	507 ,, II.
0·497 150	=log π	0·497 150	=log π

Both the foregoing solutions are true in the last figure. This last process may occasionally be of use in the absence of a seven-figure table; but, of course, no one would think of employing it with such a table at hand.

In all the foregoing examples I have employed the ordinary form of division in the resolving process; but it is strongly recommended, in practice, to use the short form, that in which the remainders only of the several partial divisions are set down. This method, while it saves the writing of many figures, and imparts great compactness to the work, will also, after a little practice, be found to be really easier than the more usual and much more lengthy operation. In illustration I subjoin, worked as here directed, the portion of *Ex.* 2 to which this remark has reference:—

3 141 592 653 590	÷3
1·047)197 551 197	188
92 85	
9 091	
1·047 19$\cancel{7}$)715 197	682
86 879	
3 103	
1 009	964
67	
4	

Problem II.

To find the number corresponding to any given logarithm, to twelve or thirteen places.

First. When the mantissa of the given logarithm is less than log 2, =·301,029,995,664.

1. Neglecting the index, cut down, or make up by the annexation of ciphers, the given logarithm to twelve places.

2. Decompose the logarithm thus modified, by successive subtraction of values taken from the several columns of the principal table, in order. The values thus taken from the table are to be in each case the greatest contained in the column in use, which does not exceed the number from which it has to be subtracted; and the arguments corresponding to them are to be placed in the margin. The result will be, that the given logarithm will be exhausted by a single entry in each column.

3. Proceed as follows with the four triads of figures placed in the margin:—To the first triad prefix unity followed by the decimal point, and set down the number thus formed as a multiplicand. Opposite, leaving space between for three more periods of three figures each, place the second triad, drawing a vertical line for a separator to the left of it. Now multiply by the triad just set down, commencing with the left hand figure, observing that the last (the right hand) figure of the first partial product will fall in the seventh decimal place, and that each successive product will fall one place to the right of that which precedes it. Add together the multiplicand and the three partial products.

4. Point off the first six figures of this sum for the effective new multiplicand, and opposite place the third triad for a multiplier. Multiply as before, taking account of the carriage from the figures cut off, and curtailing the multiplicand by one figure at each step. The last figure of each product will now fall in the twelfth place. Opposite the last formed product place the fourth triad, and continue the multiplication till all the figures of this triad have been used. The sum of the multiplicand and the partial products, when pointed in accordance with the index of the given logarithm, will be the number required.

Secondly. When the mantissa of the given logarithm is not less than log 2.

1. Prepare the given logarithm by subtracting from it any value in the auxiliary table that will leave a remainder less than ·3010299 . . . , and proceed with this remainder as above directed.

2. Multiply the sum which in the previous case formed the final result by the number corresponding to the logarithm, or divide it by the number corresponding to the co-logarithm, employed in the preparation, as the case may be; and the product, properly pointed, will be the number required.

It will now be found that here, in strict analogy with what obtains in the converse case of Problem I., every logarithm, whether it exceed or fall short of log 2 (indices being disregarded), admits of the use of four or five preparing logarithms. This is convenient, on account of the facility it affords for verification.

I now offer a few examples.

Ex. 11.—Find the anti-logarithm of 2·128,768,608,398. (See *Ex.* 1.)

128 768 608 398	345
128 722 284 338	
46 324 060	106
46 032 775	
291 285	670
290 977	
308	709
308	
1·345	106
134 5	
8 070	
1·345 14$\cancel{2}$ 570	670
807 086	
94 160	709
942	
12	
1·345 143 472 200	
134·514 347 220	=No. req.

128 768 608 398	log 9
954 242 509 439	
174 526 098 959	494
174 350 597 479	
175 501 480	404
175 419 538	
81 942	188
81 647	
295	679
295	
1·494	404
597 6	
5 976	
1·494 60$\cancel{3}$ 576	188
149 460	
119 568	
11 957	679
897	
105	
13	
1·494 603 858 000	×9
13 451 434 722 000	
134·514 347 220	=No. req.

The above, as indicated, is the converse of *Ex.* 1. Two solutions are given, and both are true in the last figure. By attention

to the precepts no difficulty will be found in following out the operations. In each, the first portion shows the decomposition of the given logarithm into a series of tabular logarithms, and the second exhibits the multiplication of the corresponding numbers.

It will be observed, that every logarithm and every partial product which appears in each of the operations here, appears also in the corresponding converse operation of *Ex.* 1. The reason of this is sufficiently obvious.

Ex. 12.—Given $\log \pi = 0{\cdot}497{,}149{,}872{,}694$; to find π.

497 149 872 694	log 3
477 121 254 720	
20 028 617 974	047
19 946 681 679	
81 936 295	188
81 639 689	
296 606	682
296 189	
417	960
417	
—	
1·047	188
104 7	
83 76	
8 376	
1·047 19~~6~~ 836	682
628 318	
83 776	
2 094	960
942	
63	
1·047 197 551 193	×3
3·141 592 653 579	$=\pi$

497 149 872 694	colog 4
397 940 008 672	
99 209 864 022	256
98 989 639 401	
220 224 621	507
220 131 504	
93 117	214
92 939	
178	410
178	
—	
1·256	507
628 0	
8 792	
1·256 63~~6~~ 792	214
251 327	
12 566	
5 027	410
503	
13	
1·256 637 061 436	÷4
314 159 265 359 0	
3·141 592 653 590	$=\pi$

Two solutions are given, as usual. In the first, the same concurrence as in the converse operation, of five similarly affected logarithms, takes place; and the consequence is, an error of 11 in the twelfth and thirteenth places of the resulting number. If we restrict ourselves to twelve places, the error will be only 1 in the last place. The second solution is true in the thirteenth place. In general, in the results of Problem II. we shall have the twelfth

place true, and an approximation to the truth, more or less close, in the thirteenth.

Ex. 13.—Given M=·434,294,481,903; required the corresponding number.

434 294 481 903	log 2
301 029 995 664	
133 264 486 239	359
133 219 456 732	
45 029 507	103
44 730 028	
299 479	689
299 229	
250	576
250	
1·359	103
135 9	
4 077	
1·359 13$\cancel{9}$ 977	689
815 484	
108 731	
12 232	576
680	
95	
8	
1·359 140 914 230	$\times 2$
2·718 281 828 460	$=\epsilon$

434 294 481 903	colog 4
397 940 008 672	
36 354 473 231	087
36 229 544 086	
124 929 145	287
124 624 634	
304 511	701
304 440	
71	163
71	
1·087	287
217 4	
86 96	
7 609	
1·087 31$\cancel{1}$ 969	701
761 118	
1 087	163
109	
65	
3	
1·087 312 731 382	$\div 4$
271 828 182 845 5	
2·718 281 828 455	$=\epsilon$

The results of the two solutions here differ by 5 in the thirteenth place. The last two figures should be 59.

Ex. 14.—Required the one thousandth root of 2.

The number here required is the value of $2^{\cdot 001}$, the logarithm of which is

$$(\log 2)\times \cdot 001 = 0\cdot 000,301,029,996;$$

and the operation is as follows:—

000 301 029 996	693 Col. II.
300 861 839	
168 157	387 „ III.
168 072	
85	196 „ IV.
85	
1·000 69~~3~~	387
300 208	
80 056	
7 005	196
100	
90	
6	
1·000 693 387 465	$=2^{\cdot 001}$

000 301 029 996	log 9
954 242 509 439	
46 058 520 557	111
45 714 058 941	
344 461 616	793
344 259 043	
202 573	466
202 381	
192	442
192	
1·111	793
777 7	
99 99	
3 333	
1·111 88~~1~~ 023	466
444 752	
66 713	
6 671	442
445	
44	
2	
1 111 881 541 627	$\times 9$
10 006 933 874 643	
1·000 693 387 464	$=2^{\cdot 001}$

Here, as shown in the first solution, there is no need for an entry in the Auxiliary Table, nor in Col. I. of the principal table. In consequence, many figures are saved. In the second solution, added for verification, log 9 is used as a preparing logarithm. (See *Ex.* 7.)

Ex. 15.—It is required to evaluate 9^{9^9}, which is the greatest number that can be expressed by three figures in the Arabic notation.

The number here to be determined is, that power of 9 whose exponent is 9^9, or 387,420,489. Its logarithm consequently is

$$(\log 9) \times 387{,}420{,}489 = 369{,}693{,}099 \cdot 631{,}570{,}358{,}743;^*$$

and the operation is as follows:—

* It is hardly necessary to mention that to obtain this product true to the last figure, it is requisite to use log 9 to about twenty-three places. I take it to this extent from Callet.

631 570 358 743	log 4	631 570 358 743	colog 3
602 059 991 328		522 878 745 280	
29 510 367 415	070	108 691 613 463	284
29 383 777 685		108 565 023 733	
126 589 730	291	126 589 730	291
126 361 310		126 361 310	
228 420	525	228 420	525
228 005		228 005	
415	956	415	956
415		415	
1·070	291	1·284	291
214 0		256 8	
96 30		115 56	
1 070		1 284	
1·070 31~~1~~ 370	525	1·284 37~~3~~ 644	525
535 156		642 187	
21 406		25 687	
5 352	956	6 422	956
963		1 156	
54		64	
6		8	
1·070 311 932 937	×4	1·284 374 319 524	÷3
4 281 247 731 748	=No. req.	428 124 773 174 7	=No. req.

It thus appears that the required number is an integer consisting of 369,693,100 figures, the first thirteen of which are 428,124,773,174,8. If this number were written down in a line, allowing one-tenth of an inch to each figure, it would extend 583 miles 845 yards and 10 inches, or about as far as from London to Inverness; and to write it down, working day and night, at the rate of two figures per second, would occupy nearly six years.*

Ex. 16.—Given $\log \pi = 0{\cdot}497{,}149{,}873$; required π to nine or ten places.

* The foregoing example was proposed by the late Mr. Frend, who succeeded, by considerations independent of the theory of logarithms, in assigning a portion of the required number.

497 149 873	log 3
477 121 255	
20 028 618	047
19 946 682	
81 936	188
81 640	
296	682
296	
1·047	188
104 7	
83 76	
8 376	682
628	
84	
2	
1 047 197 550	× 3
3·141 592 650	$=\pi$

497 149 873	colog 4
397 940 009	
99 209 864	256
98 989 639	
220 225	507
220 132	
93	214
93	
1·256	507
628 0	
8 792	214
251	
13	
5	
1 256 637 061	÷4
314 159 265	2
3·141 592 652	$=\pi$

In consequence of the restriction in the number of places required, the operations here are somewhat simplified. The logarithms are taken out to only nine places; and in the second part of the process the necessity for the formation of a new multiplicand does not arise.

Ex. 17.—Required π to six or seven places.

497 150	log 3
477 121	
20 029	047
19 947	
82	188
82	
1·04~~7~~	188
105	
84	
8	
1 047 197	× 3
3·141 591	$=\pi$

497 150	colog 4
397 940	
99 210	256
98 990	
220	507
220	
1·25~~6~~	507
628	
9	
1 256 637	÷4
314 159	2
3·141 592	$=\pi$

This example affords further illustration of the remarks made upon the one preceding.

Before leaving this problem I shall exhibit a somewhat modified form of the latter part of the operation. The form in question, as applied to the first solution of *Ex.* 12, is as follows:—

1·047	×3
3·141	188
314 1	
251 28	
25 128	
3·141 59~~0~~ 508	682
1 884 954	
251 327	
6 283	960
2 827	
188	
3·141 592 653 579	

The modification, it will be seen, consists in using up, at the beginning of the process instead of at the end, the number corresponding to the preparing logarithm. The writing of a few figures is saved. I do not know that in any other respect the modified process is either better or worse than the one previously employed.

In another paper I shall give demonstrations of the foregoing rules, and also an historical account of the method I have sought to develop.

Auxiliary Table.

Logarithms.		Co-logarithms.	
2	301 029 995 664	2	698 970 004 336
3	477 121 254 720	3	522 878 745 280
4	602 059 991 328	4	397 940 008 672
5	698 970 004 336	5	301 029 995 664
6	778 151 250 384	6	221 848 749 616
7	845 098 040 014	7	154 901 959 986
8	903 089 986 992	8	096 910 013 008
9	954 242 509 439	9	045 757 490 561

n.	I.	II.	III.		I.	II.	III.
000	000 000 000 000	000 000 000	000 000	050	021 189 299 070	021 714 181	021 715
1	000 434 077 479	000 434 294	000 434	1	021 602 716 028	022 148 454	022 149
2	000 867 721 531	000 868 588	000 869	2	022 015 739 818	022 582 726	022 583
3	001 300 933 020	001 302 881	001 303	3	022 428 371 185	023 016 998	023 018
4	001 733 712 809	001 737 174	001 737	4	022 840 610 877	023 451 269	023 452
5	002 166 061 757	002 171 467	002 171	5	023 252 459 634	023 885 540	023 886
6	002 597 980 720	002 605 759	002 606	6	023 663 918 198	024 319 810	024 320
7	003 029 470 554	003 040 051	003 040	7	024 074 987 307	024 754 080	024 755
8	003 460 532 110	003 474 342	003 474	8	024 485 667 699	025 188 349	025 189
9	003 891 166 237	003 908 633	003 909	9	024 895 960 107	025 622 619	025 623
010	004 321 373 783	004 342 923	004 343	060	025 305 865 265	026 056 887	026 058
1	004 751 155 591	004 777 213	004 777	1	025 715 383 901	026 491 155	026 492
2	005 180 512 504	005 211 503	005 212	2	026 124 516 745	026 925 423	026 926
3	005 609 445 360	005 645 792	005 646	3	026 533 264 523	027 359 691	027 361
4	006 037 954 997	006 080 080	006 080	4	026 941 627 959	027 793 957	027 795
5	006 466 042 249	006 514 368	006 514	5	027 349 607 775	028 228 224	028 229
6	006 893 707 948	006 948 656	006 949	6	027 757 204 691	028 662 490	028 663
7	007 320 952 923	007 382 943	007 383	7	028 164 419 424	029 096 756	029 098
8	007 747 778 001	007 817 230	007 817	8	028 571 252 693	029 531 021	029 532
9	008 174 184 006	008 251 517	008 252	9	028 977 705 209	029 965 285	029 966
020	008 600 171 762	008 685 803	008 686	070	029 383 777 685	030 399 550	030 401
1	009 025 742 087	009 120 088	009 120	1	029 789 470 832	030 833 814	030 835
2	009 450 895 799	009 554 374	009 554	2	030 194 785 357	031 268 077	031 269
3	009 875 633 712	009 988 658	009 989	3	030 599 721 966	031 702 340	031 703
4	010 299 956 640	010 422 942	010 423	4	031 004 281 364	032 136 603	032 138
5	010 723 865 392	010 857 226	010 857	5	031 408 464 252	032 570 865	032 572
6	011 147 360 776	011 291 510	011 292	6	031 812 271 330	033 005 126	033 006
7	011 570 443 597	011 725 793	011 726	7	032 215 703 298	033 439 388	033 441
8	011 993 114 659	012 160 075	012 160	8	032 618 760 851	033 873 649	033 875
9	012 415 374 762	012 594 357	012 595	9	033 021 444 683	034 307 909	034 309
030	012 837 224 705	013 028 639	013 029	080	033 423 755 487	034 742 169	034 744
1	013 258 665 284	013 462 920	013 463	1	033 825 693 953	035 176 428	035 178
2	013 679 697 291	013 897 201	013 897	2	034 227 260 771	035 610 687	035 612
3	014 100 321 520	014 331 481	014 332	3	034 628 456 625	036 044 946	036 046
4	014 520 538 758	014 765 761	014 766	4	035 029 282 202	036 479 204	036 481
5	014 940 349 793	015 200 041	015 200	5	035 429 738 185	036 913 462	036 915
6	015 359 755 409	015 634 320	015 635	6	035 829 825 253	037 347 720	037 349
7	015 778 756 389	016 068 599	016 069	7	036 229 544 086	037 781 976	037 784
8	016 197 353 512	016 502 877	016 503	8	036 628 895 362	038 216 233	038 218
9	016 615 547 557	016 937 155	016 937	9	037 027 879 756	038 650 489	038 652
040	017 033 339 299	017 371 432	017 372	090	037 426 497 941	039 084 745	039 087
1	017 450 729 511	017 805 709	017 806	1	037 824 750 588	039 519 000	039 521
2	017 867 718 964	018 239 985	018 240	2	038 222 638 369	039 953 255	039 955
3	018 284 308 427	018 674 261	018 675	3	038 620 161 950	040 387 509	040 389
4	018 700 498 666	019 108 537	019 109	4	039 017 321 997	040 821 763	040 824
5	019 116 290 447	019 542 812	019 543	5	039 414 119 176	041 256 016	041 258
6	019 531 684 531	019 977 087	019 978	6	039 810 554 148	041 690 269	041 692
7	019 946 681 679	020 411 361	020 412	7	040 206 627 575	042 124 522	042 127
8	020 361 282 648	020 845 635	020 846	8	040 602 340 114	042 558 774	042 561
9	020 775 488 194	021 279 908	021 280	9	040 997 692 423	042 993 026	042 995

n.	I.	II.	III.		I.	II.	III.
100	041 392 685 158	043 427 277	043 429	150	060 697 840 354	065 139 287	065 144
1	041 787 318 972	043 861 528	043 864	1	061 075 323 630	065 573 516	065 578
2	042 181 594 516	044 295 778	044 298	2	061 452 479 087	066 007 745	066 013
3	042 575 512 440	044 730 028	044 732	3	061 829 307 295	066 441 973	066 447
4	042 969 073 393	045 164 278	045 167	4	062 205 808 820	066 876 201	066 881
5	043 362 278 021	045 598 527	045 601	5	062 581 984 228	067 310 428	067 316
6	043 755 126 969	046 032 775	046 035	6	062 957 834 085	067 744 655	067 750
7	044 147 620 879	046 467 024	046 470	7	063 333 358 952	068 178 882	068 184
8	044 539 760 392	046 901 271	046 904	8	063 708 559 391	068 613 108	068 619
9	044 931 546 149	047 335 519	047 338	9	064 083 435 964	069 047 334	069 053
110	045 322 978 787	047 769 766	047 772	160	064 457 989 227	069 481 559	069 487
1	045 714 058 941	048 204 012	048 207	1	064 832 219 739	069 915 784	069 921
2	046 104 787 246	048 638 258	048 641	2	065 206 128 054	070 350 008	070 356
3	046 495 164 335	049 072 504	049 075	3	065 579 714 728	070 784 232	070 790
4	046 885 190 838	049 506 749	049 510	4	065 952 980 314	071 218 455	071 224
5	047 274 867 384	049 940 994	049 944	5	066 325 925 362	071 652 678	071 659
6	047 664 194 602	050 375 238	050 378	6	066 698 550 423	072 086 901	072 093
7	048 053 173 116	050 809 482	050 812	7	067 070 856 045	072 521 123	072 527
8	048 441 803 550	051 243 726	051 247	8	067 442 842 776	072 955 345	072 961
9	048 830 086 528	051 677 969	051 681	9	067 814 511 162	073 389 566	073 396
120	049 218 022 670	052 112 211	052 115	170	068 185 861 746	073 823 787	073 830
1	049 605 612 595	052 546 453	052 550	1	068 556 895 072	074 258 008	074 264
2	049 992 856 920	052 980 695	052 984	2	068 927 611 682	074 692 228	074 699
3	050 379 756 261	053 414 936	053 418	3	069 298 012 116	075 126 447	075 133
4	050 766 311 233	053 849 177	053 853	4	069 668 096 912	075 560 666	075 567
5	051 152 522 447	054 283 418	054 287	5	070 037 866 608	075 994 885	076 002
6	051 538 390 515	054 717 658	054 721	6	070 407 321 740	076 429 103	076 436
7	051 923 916 046	055 151 897	055 155	7	070 776 462 843	076 863 321	076 870
8	052 309 099 647	055 586 136	055 590	8	071 145 290 451	077 297 539	077 304
9	052 693 941 925	056 020 375	056 024	9	071 513 805 095	077 731 755	077 739
130	053 078 443 483	056 454 613	056 458	180	071 882 007 306	078 165 972	078 173
1	053 462 604 925	056 888 851	056 893	1	072 249 897 614	078 600 188	078 607
2	053 846 426 852	057 323 088	057 327	2	072 617 476 545	079 034 404	079 042
3	054 229 909 863	057 757 325	057 761	3	072 984 744 628	079 468 619	079 476
4	054 613 054 557	058 191 562	058 195	4	073 351 702 387	079 902 834	079 910
5	054 995 861 529	058 625 798	058 630	5	073 718 350 346	080 337 048	080 344
6	055 378 331 375	059 060 034	059 064	6	074 084 689 028	080 771 262	080 779
7	055 760 464 688	059 494 269	059 498	7	074 450 718 955	081 205 476	081 213
8	056 142 262 059	059 928 504	059 933	8	074 816 440 645	081 639 689	081 647
9	056 523 724 079	060 362 738	060 367	9	075 181 854 619	082 073 901	082 082
140	056 904 851 336	060 796 972	060 801	190	075 546 961 393	082 508 114	082 516
1	057 285 644 418	061 231 205	061 236	1	075 911 761 483	082 942 325	082 950
2	057 666 103 910	061 665 438	061 670	2	076 276 255 404	083 376 537	083 385
3	058 046 230 395	062 099 671	062 104	3	076 640 443 670	083 810 748	083 819
4	058 426 024 457	062 533 903	062 538	4	077 004 326 793	084 244 958	084 253
5	058 805 486 676	062 968 135	062 973	5	077 367 905 284	084 679 168	084 687
6	059 184 617 631	063 402 366	063 407	6	077 731 179 652	085 113 378	085 122
7	059 563 417 901	063 836 597	063 841	7	078 094 150 406	085 547 587	085 556
8	059 941 888 062	064 270 827	064 276	8	078 456 818 053	085 981 795	085 990
9	060 320 028 688	064 705 057	064 710	9	078 819 183 099	086 416 004	086 425

n.	I.	II.	III.		I.	II.	III.
200	079 181 246 048	086 850 212	086 859	250	096 910 013 008	108 560 051	108 574
1	079 543 007 403	087 284 419	087 293	1	097 257 309 693	108 994 237	109 008
2	079 904 467 667	087 718 626	087 727	2	097 604 328 874	109 428 422	109 442
3	080 265 627 340	088 152 833	088 162	3	097 951 070 994	109 862 607	109 876
4	080 626 486 922	088 587 039	088 596	4	098 297 536 495	110 296 791	110 311
5	080 987 046 911	089 021 244	089 030	5	098 643 725 817	110 730 975	110 745
6	081 347 307 804	089 455 450	089 465	6	098 989 639 401	111 165 159	111 179
7	081 707 270 097	089 889 654	089 899	7	099 335 277 686	111 599 342	111 614
8	082 066 934 285	090 323 859	090 333	8	099 680 641 109	112 033 525	112 048
9	082 426 300 861	090 758 063	090 768	9	100 025 730 108	112 467 707	112 482
210	082 785 370 316	091 192 266	091 202	260	100 370 545 118	112 901 889	112 917
1	083 144 143 143	091 626 469	091 636	1	100 715 086 573	113 336 070	113 351
2	083 502 619 830	092 060 672	092 070	2	101 059 354 908	113 770 251	113 785
3	083 860 800 867	092 494 874	092 505	3	101 403 350 555	114 204 432	114 219
4	084 218 686 739	092 929 076	092 939	4	101 747 073 946	114 638 612	114 654
5	084 576 277 934	093 363 277	093 373	5	102 090 525 512	115 072 791	115 088
6	084 933 574 937	093 797 478	093 808	6	102 433 705 681	115 506 970	115 522
7	085 290 578 230	094 231 679	094 242	7	102 776 614 883	115 941 149	115 957
8	085 647 288 297	094 665 879	094 676	8	103 119 253 546	116 375 328	116 391
9	086 003 705 618	095 100 078	095 110	9	103 461 622 095	116 809 505	116 825
220	086 359 830 675	095 534 278	095 545	270	103 803 720 956	117 243 683	117 259
1	086 715 663 945	095 968 476	095 979	1	104 145 550 554	117 677 860	117 694
2	087 071 205 907	096 402 675	096 413	2	104 487 111 312	118 112 037	118 128
3	087 426 457 036	096 836 873	096 848	3	104 828 403 654	118 546 213	118 562
4	087 781 417 810	097 271 070	097 282	4	105 169 427 999	118 980 388	118 997
5	088 136 088 701	097 705 267	097 716	5	105 510 184 770	119 414 564	119 431
6	088 490 470 182	098 139 464	098 151	6	105 850 674 385	119 848 739	119 865
7	088 844 562 727	098 573 660	098 585	7	106 190 897 263	120 282 913	120 300
8	089 198 366 805	099 007 855	099 019	8	106 530 853 822	120 717 087	120 734
9	089 551 882 886	099 442 051	099 453	9	106 870 544 479	121 151 261	121 168
230	089 905 111 439	099 876 246	099 888	280	107 209 969 648	121 585 434	121 602
1	090 258 052 931	100 310 440	100 322	1	107 549 129 745	122 019 606	122 037
2	090 610 707 828	100 744 634	100 756	2	107 888 025 183	122 453 779	122 471
3	090 963 076 596	101 178 827	101 191	3	108 226 656 375	122 887 951	122 905
4	091 315 159 697	101 613 021	101 625	4	108 565 023 733	123 322 122	123 340
5	091 666 957 596	102 047 213	102 059	5	108 903 127 667	123 756 293	123 774
6	092 018 470 753	102 481 405	102 493	6	109 240 968 588	124 190 463	124 208
7	092 369 699 629	102 915 597	102 928	7	109 578 546 904	124 624 634	124 642
8	092 720 644 684	103 349 789	103 362	8	109 915 863 024	125 058 803	125 077
9	093 071 306 376	103 783 979	103 796	9	110 252 917 353	125 492 972	125 511
240	093 421 685 162	104 218 170	104 231	290	110 589 710 299	125 927 141	125 945
1	093 771 781 499	104 652 360	104 665	1	110 926 242 266	126 361 310	126 380
2	094 121 595 841	105 086 550	105 099	2	111 262 513 659	126 795 477	126 814
3	094 471 128 642	105 520 739	105 534	3	111 598 524 880	127 229 645	127 248
4	094 820 380 355	105 954 928	105 968	4	111 934 276 333	127 663 812	127 683
5	095 169 351 432	106 389 116	106 402	5	112 269 768 417	128 097 979	128 117
6	095 518 042 323	106 823 304	106 836	6	112 605 001 535	128 532 145	128 551
7	095 866 453 479	107 257 491	107 271	7	112 939 976 084	128 966 311	128 985
8	096 214 585 346	107 691 678	107 705	8	113 274 692 464	129 400 476	129 420
9	096 562 438 374	108 125 865	108 139	9	113 609 151 073	129 834 641	129 854

n.	I.	II.	III.		I.	II.	III.
300	113 943 352 307	130 268 805	130 288	350	130 333 768 495	151 976 474	152 003
1	114 277 296 562	130 702 969	130 723	1	130 655 349 022	152 410 617	152 437
2	114 610 984 232	131 137 133	131 157	2	130 976 691 606	152 844 759	152 872
3	114 944 415 713	131 571 296	131 591	3	131 297 796 598	153 278 900	153 306
4	115 277 591 396	132 005 459	132 026	4	131 618 664 349	153 713 041	153 740
5	115 610 511 674	132 439 621	132 460	5	131 939 295 210	154 147 182	154 175
6	115 943 176 939	132 873 783	132 894	6	132 259 689 531	154 581 322	154 609
7	116 275 587 581	133 307 944	133 328	7	132 579 847 660	155 015 461	155 043
8	116 607 743 988	133 742 105	133 763	8	132 899 769 944	155 449 601	155 477
9	116 939 646 551	134 176 266	134 197	9	133 219 456 732	155 883 740	155 912
310	117 271 295 656	134 610 426	134 631	360	133 538 908 370	156 317 878	156 346
1	117 602 691 690	135 044 586	135 066	1	133 858 125 203	156 752 016	156 780
2	117 933 835 040	135 478 745	135 500	2	134 177 107 577	157 186 153	157 215
3	118 264 726 089	135 912 904	135 934	3	134 495 855 835	157 620 291	157 649
4	118 595 365 224	136 347 062	136 368	4	134 814 370 320	158 054 427	158 083
5	118 925 752 826	136 781 220	136 803	5	135 132 651 377	158 488 563	158 517
6	119 255 889 278	137 215 377	137 237	6	135 450 699 346	158 922 699	158 952
7	119 585 774 962	137 649 534	137 671	7	135 768 514 568	159 356 835	159 386
8	119 915 410 258	138 083 691	138 106	8	136 086 097 384	159 790 970	159 820
9	120 244 795 546	138 517 847	138 540	9	136 403 448 134	160 225 104	160 255
320	120 573 931 206	138 952 003	138 974	370	136 720 567 156	160 659 238	160 689
1	120 902 817 615	139 386 158	139 409	1	137 037 454 790	161 093 372	161 123
2	121 231 455 150	139 820 313	139 843	2	137 354 111 371	161 527 505	161 558
3	121 559 844 188	140 254 468	140 277	3	137 670 537 237	161 961 638	161 992
4	121 887 985 104	140 688 622	140 711	4	137 986 732 724	162 395 770	162 426
5	122 215 878 273	141 122 775	141 146	5	138 302 698 166	162 829 902	162 860
6	122 543 524 069	141 556 929	141 580	6	138 618 433 899	163 264 033	163 295
7	122 870 922 864	141 991 081	142 014	7	138 933 940 257	163 698 165	163 729
8	123 198 075 032	142 425 234	142 449	8	139 249 217 572	164 132 295	164 163
9	123 524 980 943	142 859 385	142 883	9	139 564 266 176	164 566 425	164 598
330	123 851 640 967	143 293 537	143 317	380	139 879 086 401	165 000 555	165 032
1	124 178 055 475	143 727 688	143 751	1	140 193 678 579	165 434 684	165 466
2	124 504 224 834	144 161 838	144 186	2	140 508 043 038	165 868 813	165 900
3	124 830 149 414	144 595 989	144 620	3	140 822 180 109	166 302 942	166 335
4	125 155 829 581	145 030 138	145 054	4	141 136 090 121	166 737 070	166 769
5	125 481 265 701	145 464 288	145 489	5	141 449 773 400	167 171 197	167 203
6	125 806 458 140	145 898 436	145 923	6	141 763 230 276	167 605 324	167 638
7	126 131 407 262	146 332 585	146 357	7	142 076 461 073	168 039 451	168 072
8	126 456 113 432	146 766 733	146 792	8	142 389 466 119	168 473 577	168 506
9	126 780 577 012	147 200 880	147 226	9	142 702 245 738	168 907 703	168 941
340	127 104 798 365	147 635 027	147 660	390	143 014 800 254	169 341 828	169 375
1	127 428 777 852	148 069 174	148 094	1	143 327 129 992	169 775 953	169 809
2	127 752 515 833	148 503 320	148 529	2	143 639 235 275	170 210 078	170 243
3	128 076 012 669	148 937 466	148 963	3	143 951 116 424	170 644 202	170 678
4	128 399 268 718	149 371 611	149 397	4	144 262 773 762	171 078 326	171 112
5	128 722 284 338	149 805 756	149 832	5	144 574 207 610	171 512 449	171 546
6	129 045 059 888	150 239 901	150 266	6	144 885 418 287	171 946 572	171 981
7	129 367 595 723	150 674 045	150 700	7	145 196 406 114	172 380 694	172 415
8	129 689 892 199	151 108 188	151 134	8	145 507 171 410	172 814 816	172 849
9	130 011 949 672	151 542 332	151 569	9	145 817 714 492	173 248 937	173 283

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400	146 128 035 678	173 683 058	173 718	450	161 368 002 235	195 388 558	195 432
1	146 438 135 286	174 117 179	174 152	1	161 667 412 438	195 822 657	195 867
2	146 748 013 631	174 551 299	174 586	2	161 966 616 364	196 256 755	196 301
3	147 057 671 028	174 985 419	175 021	3	162 265 614 298	196 690 853	196 735
4	147 367 107 794	175 419 538	175 455	4	162 564 406 523	197 124 951	197 170
5	147 676 324 241	175 853 657	175 889	5	162 862 993 322	197 559 048	197 604
6	147 985 320 684	176 287 776	176 324	6	163 161 374 977	197 993 145	198 038
7	148 294 097 435	176 721 894	176 758	7	163 459 551 770	198 427 241	198 473
8	148 602 654 806	177 156 011	177 192	8	163 757 523 982	198 861 337	198 907
9	148 910 993 109	177 590 128	177 626	9	164 055 291 893	199 295 432	199 341
410	149 219 112 655	178 024 245	178 061	460	164 352 855 784	199 729 527	199 775
1	149 527 013 754	178 458 361	178 495	1	164 650 215 934	200 163 622	200 210
2	149 834 696 716	178 892 477	178 929	2	164 947 372 622	200 597 716	200 644
3	150 142 161 849	179 326 593	179 364	3	165 244 326 125	201 031 810	201 078
4	150 449 409 461	179 760 708	179 798	4	165 541 076 722	201 465 903	201 513
5	150 756 439 860	180 194 822	180 232	5	165 837 624 690	201 899 996	201 947
6	151 063 253 354	180 628 936	180 666	6	166 133 970 305	202 334 088	202 381
7	151 369 850 247	181 063 050	181 101	7	166 430 113 843	202 768 180	202 815
8	151 676 230 847	181 497 163	181 535	8	166 726 055 580	203 202 272	203 250
9	151 982 395 457	181 931 276	181 969	9	167 021 795 790	203 636 363	203 684
420	152 288 344 383	182 365 388	182 404	470	167 317 334 748	204 070 454	204 118
1	152 594 077 927	182 799 500	182 838	1	167 612 672 728	204 504 544	204 553
2	152 899 596 394	183 233 612	183 272	2	167 907 810 001	204 938 634	204 987
3	153 204 900 084	183 667 723	183 707	3	168 202 746 843	205 372 723	205 421
4	153 509 989 301	184 101 833	184 141	4	168 497 483 523	205 806 812	205 856
5	153 814 864 345	184 535 944	184 575	5	168 792 020 314	206 240 901	206 290
6	154 119 525 516	184 970 053	185 009	6	169 086 357 487	206 674 989	206 724
7	154 423 973 115	185 404 163	185 444	7	169 380 495 312	207 109 076	207 158
8	154 728 207 440	185 838 272	185 878	8	169 674 434 059	207 543 163	207 593
9	155 032 228 791	186 272 380	186 312	9	169 968 173 997	207 977 250	208 027
430	155 336 037 465	186 706 488	186 747	480	170 261 715 395	208 411 337	208 461
1	155 639 633 760	187 140 596	187 181	1	170 555 058 521	208 845 422	208 896
2	155 943 017 972	187 574 703	187 615	2	170 848 203 643	209 279 508	209 330
3	156 246 190 397	188 008 810	188 049	3	171 141 151 028	209 713 593	209 764
4	156 549 151 332	188 442 916	188 484	4	171 433 900 943	210 147 678	210 198
5	156 851 901 070	188 877 022	188 918	5	171 726 453 653	210 581 762	210 633
6	157 154 439 906	189 311 127	189 352	6	172 018 809 425	211 015 846	211 067
7	157 456 768 134	189 745 232	189 787	7	172 310 968 522	211 449 929	211 501
8	157 758 886 047	190 179 337	190 221	8	172 602 931 210	211 884 012	211 936
9	158 060 793 937	190 613 441	190 655	9	172 894 697 752	212 318 094	212 370
440	158 362 492 095	191 047 545	191 090	490	173 186 268 412	212 752 176	212 804
1	158 663 980 814	191 481 648	191 524	1	173 477 643 453	213 186 258	213 239
2	158 965 260 383	191 915 751	191 958	2	173 768 823 137	213 620 339	213 673
3	159 266 331 093	192 349 853	192 392	3	174 059 807 725	214 054 419	214 107
4	159 567 193 234	192 783 955	192 827	4	174 350 597 479	214 488 500	214 541
5	159 867 847 093	193 218 057	193 261	5	174 641 192 660	214 922 580	214 976
6	160 168 292 959	193 652 158	193 695	6	174 931 593 528	215 356 659	215 410
7	160 468 531 119	194 086 258	194 130	7	175 221 800 343	215 790 738	215 844
8	160 768 561 861	194 520 359	194 564	8	175 511 813 363	216 224 816	216 279
9	161 068 385 471	194 954 458	194 998	9	175 801 632 848	216 658 895	216 713

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500	176 091 259 056	217 092 972	217 147	550	190 331 698 170	238 796 302	238 862
1	176 380 692 243	217 527 049	217 581	1	190 611 797 814	239 230 358	239 296
2	176 669 932 668	217 961 126	218 016	2	190 891 716 922	239 664 413	239 730
3	176 958 980 587	218 395 203	218 450	3	191 171 455 729	240 098 467	240 165
4	177 247 836 256	218 829 279	218 884	4	191 451 014 465	240 532 522	240 599
5	177 536 499 930	219 263 354	219 319	5	191 730 393 363	240 966 575	241 033
6	177 824 971 865	219 697 429	219 753	6	192 009 592 654	241 400 629	241 468
7	178 113 252 315	220 131 504	220 187	7	192 288 612 568	241 834 682	241 902
8	178 401 341 534	220 565 578	220 622	8	192 567 453 337	242 268 734	242 336
9	178 689 239 776	220 999 652	221 056	9	192 846 115 189	242 702 786	242 771
510	178 976 947 293	221 433 725	221 490	560	193 124 598 354	243 136 838	243 205
1	179 264 464 339	221 867 798	221 924	1	193 402 903 062	243 570 889	243 639
2	179 551 791 165	222 301 870	222 359	2	193 681 029 541	244 004 940	244 073
3	179 838 928 023	222 735 942	222 793	3	193 958 978 019	244 438 990	244 508
4	180 125 875 164	223 170 014	223 227	4	194 236 748 724	244 873 040	244 942
5	180 412 632 838	223 604 085	223 662	5	194 514 341 882	245 307 090	245 376
6	180 699 201 296	224 038 156	224 096	6	194 791 757 722	245 741 139	245 811
7	180 985 580 787	224 472 226	224 530	7	195 068 996 469	246 175 187	246 245
8	181 271 771 559	224 906 296	224 964	8	195 346 058 348	246 609 235	246 679
9	181 557 773 863	225 340 365	225 399	9	195 622 943 587	247 043 283	247 113
520	181 843 587 945	225 774 434	225 833	570	195 899 652 409	247 477 330	247 548
1	182 129 214 053	226 208 503	226 267	1	196 176 185 040	247 911 377	247 982
2	182 414 652 435	226 642 571	226 702	2	196 452 541 703	248 345 424	248 416
3	182 699 903 336	227 076 639	227 136	3	196 728 722 623	248 779 470	248 851
4	182 984 967 004	227 510 706	227 570	4	197 004 728 023	249 213 515	249 285
5	183 269 843 683	227 944 773	228 005	5	197 280 558 126	249 647 560	249 719
6	183 554 533 619	228 378 839	228 439	6	197 556 213 154	250 081 605	250 154
7	183 839 037 056	228 812 905	228 873	7	197 831 693 329	250 515 649	250 588
8	184 123 354 240	229 246 971	229 307	8	198 106 998 873	250 949 693	251 022
9	184 407 485 412	229 681 036	229 742	9	198 382 130 008	251 383 736	251 456
530	184 691 430 818	230 115 100	230 176	580	198 657 086 954	251 817 779	251 891
1	184 975 190 698	230 549 165	230 610	1	198 931 869 932	252 251 822	252 325
2	185 258 765 297	230 983 228	231 045	2	199 206 479 162	252 685 864	252 759
3	185 542 154 854	231 417 292	231 479	3	199 480 914 862	253 119 906	253 194
4	185 825 359 613	231 851 355	231 913	4	199 755 177 253	253 553 947	253 628
5	186 108 379 813	232 285 417	232 347	5	200 029 266 554	253 987 988	254 062
6	186 391 215 695	232 719 479	232 782	6	200 303 182 982	254 422 028	254 496
7	186 673 867 500	233 153 541	233 216	7	200 576 926 755	254 856 068	254 931
8	186 956 335 465	233 587 602	233 650	8	200 850 498 091	255 290 107	255 365
9	187 238 619 831	234 021 663	234 085	9	201 123 897 207	255 724 146	255 799
540	187 520 720 836	234 455 723	234 519	590	201 397 124 320	256 158 185	256 234
1	187 802 638 718	234 889 783	234 953	1	201 670 179 647	256 592 223	256 668
2	188 084 373 715	235 323 842	235 388	2	201 943 063 402	257 026 261	257 102
3	188 365 926 063	235 757 901	235 822	3	202 215 775 801	257 460 298	257 537
4	188 647 296 000	236 191 960	236 256	4	202 488 317 060	257 894 335	257 971
5	188 928 483 761	236 626 018	236 690	5	202 760 687 393	258 328 372	258 405
6	189 209 489 582	237 060 076	237 125	6	203 032 887 015	258 762 408	258 839
7	189 490 313 699	237 494 133	237 559	7	203 304 916 138	259 196 443	259 274
8	189 770 956 347	237 928 190	237 993	8	203 576 774 978	259 630 478	259 708
9	190 051 417 759	238 362 246	238 428	9	203 848 463 746	260 064 513	260 142

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600	204 119 982 656	260 498 547	260 577	650	217 483 944 214	282 199 708	282 291
1	204 391 331 919	260 932 581	261 011	1	217 747 073 263	282 633 720	282 726
2	204 662 511 748	261 366 615	261 445	2	218 010 042 984	283 067 732	283 160
3	204 933 522 354	261 800 648	261 879	3	218 272 853 571	283 501 743	283 594
4	205 204 363 948	262 234 680	262 314	4	218 535 505 217	283 935 754	284 028
5	205 475 036 741	262 668 712	262 748	5	218 797 998 112	284 369 765	284 463
6	205 745 540 943	263 102 744	263 182	6	219 060 332 449	284 803 775	284 897
7	206 015 876 763	263 536 775	263 617	7	219 322 508 419	285 237 784	285 331
8	206 286 044 412	263 970 806	264 051	8	219 584 526 214	285 671 793	285 766
9	206 556 044 099	264 404 836	264 485	9	219 846 386 024	286 105 802	286 200
610	206 825 876 032	264 838 866	264 920	660	220 108 088 040	286 539 810	286 634
1	207 095 540 419	265 272 896	265 354	1	220 369 632 451	286 973 818	287 069
2	207 365 037 469	265 706 925	265 788	2	220 631 019 448	287 407 826	287 503
3	207 634 367 389	266 140 954	266 222	3	220 892 249 220	287 841 832	287 937
4	207 903 530 386	266 574 982	266 657	4	221 153 321 955	288 275 839	288 371
5	208 172 526 667	267 009 010	267 091	5	221 414 237 842	288 709 845	288 806
6	208 441 356 439	267 443 037	267 525	6	221 674 997 071	289 143 851	289 240
7	208 710 019 906	267 877 064	267 960	7	221 935 599 828	289 577 856	289 674
8	208 978 517 276	268 311 090	268 394	8	222 196 046 302	290 011 861	290 109
9	209 246 848 753	268 745 116	268 828	9	222 456 336 679	290 445 865	290 543
620	209 515 014 543	269 179 142	269 262	670	222 716 471 148	290 879 869	290 977
1	209 783 014 849	269 613 167	269 697	1	222 976 449 893	291 313 872	291 411
2	210 050 849 875	270 047 192	270 131	2	223 236 273 103	291 747 876	291 846
3	210 318 519 826	270 481 216	270 565	3	223 495 940 962	292 181 878	292 280
4	210 586 024 905	270 915 240	271 000	4	223 755 453 657	292 615 880	292 714
5	210 853 365 315	271 349 263	271 434	5	224 014 811 373	293 049 882	293 149
6	211 120 541 258	271 783 286	271 868	6	224 274 014 294	293 483 883	293 583
7	211 387 552 937	272 217 309	272 303	7	224 533 062 606	293 917 884	294 017
8	211 654 400 553	272 651 331	272 737	8	224 791 956 493	294 351 885	294 452
9	211 921 084 309	273 085 353	273 171	9	225 050 696 138	294 785 885	294 886
630	212 187 604 404	273 519 374	273 605	680	225 309 281 726	295 219 884	295 320
1	212 453 961 040	273 953 395	274 040	1	225 567 713 439	295 653 883	295 754
2	212 720 154 418	274 387 415	274 474	2	225 825 991 462	296 087 882	296 189
3	212 986 184 737	274 821 435	274 908	3	226 084 115 976	296 521 880	296 623
4	213 252 052 196	275 255 455	275 343	4	226 342 087 164	296 955 878	297 057
5	213 517 756 996	275 689 474	275 777	5	226 599 905 207	297 389 876	297 492
6	213 783 299 335	276 123 493	276 211	6	226 857 570 289	297 823 873	297 926
7	214 048 679 412	276 557 511	276 645	7	227 115 082 589	298 257 869	298 360
8	214 313 897 424	276 991 529	277 080	8	227 372 442 290	298 691 865	298 795
9	214 578 953 570	277 425 546	277 514	9	227 629 649 571	299 125 861	299 229
640	214 843 848 048	277 859 563	277 948	690	227 886 704 614	299 559 856	299 663
1	215 108 581 053	278 293 579	278 383	1	228 143 607 598	299 993 851	300 097
2	215 373 152 783	278 727 595	278 817	2	228 400 358 703	300 427 845	300 532
3	215 637 563 435	279 161 611	279 251	3	228 656 958 109	300 861 839	300 966
4	215 901 813 204	279 595 626	279 686	4	228 913 405 995	301 295 833	301 400
5	216 165 902 286	280 029 641	280 120	5	229 169 702 539	301 729 826	301 835
6	216 429 830 876	280 463 655	280 554	6	229 425 847 921	302 163 819	302 269
7	216 693 599 170	280 897 669	280 988	7	229 681 842 318	302 597 811	302 703
8	216 957 207 361	281 331 683	281 423	8	229 937 685 908	303 031 803	303 137
9	217 220 655 645	281 765 696	281 857	9	230 193 378 869	303 465 794	303 572

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700	230 448 921 378	303 899 785	304 006	750	243 038 048 686	325 598 777	325 721
1	230 704 313 613	304 333 775	304 440	1	243 286 146 083	326 032 746	326 155
2	230 959 555 749	304 767 765	304 875	2	243 534 101 832	326 466 714	326 589
3	231 214 647 963	305 201 755	305 309	3	243 781 916 094	326 900 682	327 024
4	231 469 590 431	305 635 744	305 743	4	244 029 589 030	327 334 650	327 458
5	231 724 383 329	306 069 733	306 178	5	244 277 120 802	327 768 617	327 892
6	231 979 026 832	306 503 721	306 612	6	244 524 511 570	328 202 583	328 327
7	232 233 521 115	306 937 709	307 046	7	244 771 761 495	328 636 550	328 761
8	232 487 866 353	307 371 696	307 480	8	245 018 870 738	329 070 515	329 195
9	232 742 062 721	307 805 683	307 915	9	245 265 839 457	329 504 481	329 629
710	232 996 110 392	308 239 670	308 349	760	245 512 667 814	329 938 446	330 064
1	233 250 009 541	308 673 656	308 783	1	245 759 355 967	330 372 410	330 498
2	233 503 760 341	309 107 642	309 218	2	246 005 904 076	330 806 374	330 932
3	233 757 362 966	309 541 627	309 652	3	246 252 312 299	331 240 338	331 367
4	234 010 817 587	309 975 612	310 086	4	246 498 580 796	331 674 301	331 801
5	234 264 124 379	310 409 596	310 520	5	246 744 709 724	332 108 263	332 235
6	234 517 283 513	310 843 580	310 955	6	246 990 699 242	332 542 226	332 669
7	234 770 295 161	311 277 564	311 389	7	247 236 549 507	332 976 188	333 104
8	235 023 159 495	311 711 547	311 823	8	247 482 260 677	333 410 149	333 538
9	235 275 876 687	312 145 530	312 258	9	247 727 832 910	333 844 110	333 972
720	235 528 446 908	312 579 512	312 692	770	247 973 266 362	334 278 071	334 407
1	235 780 870 328	313 013 494	313 126	1	248 218 561 190	334 712 031	334 841
2	236 033 147 118	313 447 475	313 561	2	248 463 717 551	335 145 990	335 275
3	236 285 277 448	313 881 456	313 995	3	248 708 735 601	335 579 950	335 710
4	236 537 261 489	314 315 436	314 429	4	248 953 615 496	336 013 908	336 144
5	236 789 099 409	314 749 416	314 863	5	249 198 357 391	336 447 867	336 578
6	237 040 791 379	315 183 396	315 298	6	249 442 961 443	336 881 825	337 012
7	237 292 337 567	315 617 375	315 732	7	249 687 427 805	337 315 782	337 447
8	237 543 738 143	316 051 354	316 166	8	249 931 756 634	337 749 739	337 881
9	237 794 993 274	316 485 332	316 601	9	250 175 948 084	338 183 696	338 315
730	238 046 103 129	316 919 310	317 035	780	250 420 002 309	338 617 652	338 750
1	238 297 067 875	317 353 288	317 469	1	250 663 919 463	339 051 608	339 184
2	238 547 887 681	317 787 265	317 903	2	250 907 699 701	339 485 563	339 618
3	238 798 562 714	318 221 241	318 338	3	251 151 343 175	339 919 518	340 052
4	239 049 093 140	318 655 218	318 772	4	251 394 850 040	340 353 473	340 487
5	239 299 479 127	319 089 193	319 206	5	251 638 220 448	340 787 427	340 921
6	239 549 720 840	319 523 169	319 641	6	251 881 454 553	341 221 380	341 355
7	239 799 818 447	319 957 143	320 075	7	252 124 552 506	341 655 334	341 790
8	240 049 772 113	320 391 118	320 509	8	252 367 514 460	342 089 286	342 224
9	240 299 582 003	320 825 092	320 944	9	252 610 340 567	342 523 239	342 658
740	240 549 248 283	321 259 065	321 378	790	252 853 030 980	342 957 190	343 093
1	240 798 771 117	321 693 039	321 812	1	253 095 585 849	343 391 142	343 527
2	241 048 150 672	322 127 011	322 246	2	253 338 005 326	343 825 093	343 961
3	241 297 387 110	322 560 983	322 681	3	253 580 289 562	344 259 043	344 395
4	241 546 480 597	322 994 955	323 115	4	253 822 438 708	344 692 994	344 830
5	241 795 431 295	323 428 927	323 549	5	254 064 452 914	345 126 943	345 264
6	242 044 239 370	323 862 898	323 984	6	254 306 332 331	345 560 893	345 698
7	242 292 904 983	324 296 868	324 418	7	254 548 077 109	345 994 841	346 133
8	242 541 428 298	324 730 838	324 852	8	254 789 687 397	346 428 790	346 567
9	242 789 809 479	325 164 808	325 286	9	255 031 163 346	346 862 738	347 001

n.	I.	II.	III.		I.	II.	III.
800	255 272 505 103	347 296 685	347 435	850	267 171 728 403	368 993 510	369 150
1	255 513 712 820	347 730 632	347 870	1	267 406 418 753	369 427 435	369 584
2	255 754 786 643	348 164 579	348 304	2	267 640 982 346	369 861 360	370 019
3	255 995 726 722	348 598 525	348 738	3	267 875 419 319	370 295 285	370 453
4	256 236 533 206	349 032 471	349 173	4	268 109 729 808	370 729 209	370 887
5	256 477 206 242	349 466 417	349 607	5	268 343 913 951	371 163 132	371 322
6	256 717 745 977	349 900 362	350 041	6	268 577 971 883	371 597 056	371 756
7	256 958 152 561	350 334 306	350 476	7	268 811 903 740	372 030 978	372 190
8	257 198 426 139	350 768 250	350 910	8	269 045 709 658	372 464 901	372 625
9	257 438 566 860	351 202 194	351 344	9	269 279 389 772	372 898 823	373 059
810	257 678 574 869	351 636 137	351 778	860	269 512 944 218	373 332 744	373 493
1	257 918 450 314	352 070 080	352 213	1	269 746 373 131	373 766 665	373 927
2	258 158 193 341	352 504 022	352 647	2	269 979 676 645	374 200 586	374 362
3	258 397 804 096	352 937 964	353 081	3	270 212 854 896	374 634 506	374 796
4	258 637 282 724	353 371 905	353 516	4	270 445 908 018	375 068 426	375 230
5	258 876 629 372	353 805 846	353 950	5	270 678 836 145	375 502 345	375 665
6	259 115 844 185	354 239 787	354 384	6	270 911 639 410	375 936 264	376 099
7	259 354 927 308	354 673 727	354 818	7	271 144 317 949	376 370 183	376 533
8	259 593 878 886	355 107 667	355 253	8	271 376 871 894	376 804 101	376 967
9	259 832 699 063	355 541 606	355 687	9	271 609 301 379	377 238 019	377 402
820	260 071 387 985	355 975 545	356 121	870	271 841 606 536	377 671 936	377 836
1	260 309 945 795	356 409 484	356 556	1	272 073 787 500	378 105 853	378 270
2	260 548 372 637	356 843 422	356 990	2	272 305 844 402	378 539 769	378 705
3	260 786 668 655	357 277 359	357 424	3	272 537 777 375	378 973 685	379 139
4	261 024 833 992	357 711 296	357 859	4	272 769 586 552	379 407 600	379 573
5	261 262 868 792	358 145 233	358 293	5	273 001 272 064	379 841 515	380 008
6	261 500 773 198	358 579 169	358 727	6	273 232 834 043	380 275 430	380 442
7	261 738 547 353	359 013 105	359 161	7	273 464 272 621	380 709 344	380 876
8	261 976 191 398	359 447 040	359 596	8	273 695 587 930	381 143 258	381 310
9	262 213 705 476	359 880 975	360 030	9	273 926 780 101	381 577 171	381 745
830	262 451 089 730	360 314 910	360 464	880	274 157 849 264	382 011 084	382 179
1	262 688 344 302	360 748 844	360 899	1	274 388 795 550	382 444 996	382 613
2	262 925 469 332	361 182 778	361 333	2	274 619 619 091	382 878 908	383 048
3	263 162 464 962	361 616 711	361 767	3	274 850 320 017	383 312 820	383 482
4	263 399 331 334	362 050 644	362 201	4	275 080 898 457	383 746 731	383 916
5	263 636 068 588	362 484 576	362 636	5	275 311 354 542	384 180 642	384 350
6	263 872 676 865	362 918 508	363 070	6	275 541 688 401	384 614 552	384 785
7	264 109 156 306	363 352 440	363 504	7	275 771 900 165	385 048 462	385 219
8	264 345 507 050	363 786 371	363 939	8	276 001 989 962	385 482 371	385 653
9	264 581 729 238	364 220 301	364 373	9	276 231 957 922	385 916 280	386 088
840	264 817 823 010	364 654 231	364 807	890	276 461 804 173	386 350 189	386 522
1	265 053 788 504	365 088 161	365 242	1	276 691 528 845	386 784 097	386 956
2	265 289 625 861	365 522 091	365 676	2	276 921 132 066	387 218 004	387 391
3	265 525 335 219	265 956 019	366 110	3	277 150 613 964	387 651 912	387 825
4	265 760 916 718	366 389 948	366 544	4	277 379 974 667	388 085 818	388 259
5	265 996 370 495	366 823 876	366 979	5	277 609 214 304	388 519 725	388 693
6	266 231 696 690	367 257 804	367 413	6	277 838 333 002	388 953 631	389 128
7	266 466 895 440	367 691 731	367 847	7	278 067 330 889	389 387 536	389 562
8	266 701 966 884	368 125 657	368 282	8	278 296 208 091	389 821 441	389 996
9	266 936 911 159	368 559 584	368 716	9	278 524 964 737	390 255 346	390 431

n.	I.	II.	III.		I.	II.	III.
900	278 753 600 953	390 689 250	390 865	950	290 034 611 363	412 383 906	412 580
1	278 982 116 865	391 123 154	391 299	1	290 257 269 395	412 817 789	413 014
2	279 210 512 601	391 557 057	391 733	2	290 479 813 331	413 251 670	413 448
3	279 438 788 287	391 990 960	392 168	3	290 702 243 288	413 685 551	413 882
4	279 666 944 048	392 424 862	392 602	4	290 924 559 383	414 119 432	414 317
5	279 894 980 012	392 858 764	393 036	5	291 146 761 732	414 553 313	414 751
6	280 122 896 302	393 292 666	393 471	6	291 368 850 452	414 987 192	415 185
7	280 350 693 046	393 726 567	393 905	7	291 590 825 658	415 421 072	415 620
8	280 578 370 368	394 160 468	394 339	8	291 812 687 467	415 854 951	416 054
9	280 805 928 394	394 594 368	394 774	9	292 034 435 995	416 288 830	416 488
910	281 033 367 248	395 028 268	395 208	960	292 256 071 356	416 722 708	416 923
1	281 260 687 055	395 462 167	395 642	1	292 477 593 668	417 156 585	417 357
2	281 487 887 940	395 896 066	396 076	2	292 699 003 044	417 590 463	417 791
3	281 714 970 027	396 329 965	396 511	3	292 920 299 600	418 024 340	418 225
4	281 941 933 441	396 763 863	396 945	4	293 141 483 451	418 458 216	418 660
5	282 168 778 305	397 197 761	397 379	5	293 362 554 711	418 892 092	419 094
6	282 395 504 743	397 631 658	397 814	6	293 583 513 496	419 325 968	419 528
7	282 622 112 878	398 065 555	398 248	7	293 804 359 919	419 759 843	419 963
8	282 848 602 835	398 499 451	398 682	8	294 025 094 095	420 193 718	420 397
9	283 074 974 735	398 933 347	399 116	9	294 245 716 138	420 627 592	420 831
920	283 301 228 704	399 367 243	399 551	970	294 466 226 162	421 061 466	421 265
1	283 527 364 862	399 801 138	399 985	1	294 686 624 279	421 495 339	421 700
2	283 753 383 333	400 235 032	400 419	2	294 906 910 605	421 929 212	422 134
3	283 979 284 238	400 668 927	400 854	3	295 127 085 252	422 363 085	422 568
4	284 205 067 702	401 102 820	401 288	4	295 347 148 334	422 796 957	423 003
5	284 430 733 845	401 536 714	401 722	5	295 567 099 962	423 230 828	423 437
6	284 656 282 789	401 970 607	402 157	6	295 786 940 252	423 664 700	423 871
7	284 881 714 655	402 404 499	402 591	7	296 006 669 314	424 098 570	424 306
8	285 107 029 567	402 838 391	403 025	8	296 226 287 261	424 532 441	424 740
9	285 332 227 644	403 272 283	403 459	9	296 445 794 206	424 966 311	425 174
930	285 557 309 008	403 706 174	403 894	980	296 665 190 262	425 400 180	425 608
1	285 782 273 779	404 140 065	404 328	1	296 884 475 539	425 834 049	426 043
2	286 007 122 079	404 573 955	404 762	2	297 103 650 149	426 267 918	426 477
3	286 231 854 029	405 007 845	405 197	3	297 322 714 205	426 701 786	426 911
4	286 456 469 747	405 441 734	405 631	4	297 541 667 818	427 135 654	427 346
5	286 680 969 355	405 875 623	406 065	5	297 760 511 099	427 569 521	427 780
6	286 905 352 972	406 309 512	406 499	6	297 979 244 159	428 003 388	428 214
7	287 129 620 719	406 743 400	406 934	7	298 197 867 110	428 437 255	428 648
8	287 353 772 715	407 177 288	407 368	8	298 416 380 061	428 871 121	429 083
9	287 577 809 079	407 611 175	407 802	9	298 634 783 124	429 304 986	429 517
940	287 801 729 930	408 045 062	408 237	990	298 853 076 410	429 738 851	429 951
1	288 025 535 388	408 478 948	408 671	1	299 071 260 027	430 172 716	430 386
2	288 249 225 572	408 912 834	409 105	2	299 289 334 088	430 606 580	430 820
3	288 472 800 600	409 346 720	409 540	3	299 507 298 700	431 040 444	431 254
4	288 696 260 590	409 780 605	409 974	4	299 725 153 976	431 474 308	431 689
5	288 919 605 662	410 214 490	410 408	5	299 942 900 023	431 908 171	432 123
6	289 142 835 932	410 648 374	410 842	6	300 160 536 951	432 342 033	432 557
7	289 365 951 520	411 082 258	411 277	7	300 378 064 871	432 775 896	432 991
8	289 588 952 543	411 516 141	411 711	8	300 595 483 890	433 209 757	433 426
9	289 811 839 118	411 950 024	412 145	9	300 812 794 118	433 643 618	433 860

A Budget of Paradoxes. By Professor De Morgan.

No. XI. 1819—1825.

(Continued from p. 48.)

The Mythological Astronomy of the ancients; part the second: or the key of Urania, the wards of which will unlock all the mysteries of antiquity. Norwich, 1823, 12mo.

A Companion to the Mythological Astronomy, &c., containing remarks on recent publications . . . . Norwich, 1824, 12mo.

A new Theory of the Earth and of planetary motion; in which it is demonstrated that the Sun is vicegerent of his own system. Norwich, 1825, 12mo.

The analyzation of the writings of the Jews, so far as they are found to have any connexion with the sublime science of astronomy. [This is pp. 97–180 of some other work, being all I have seen.]

These works are all by Sampson Arnold Mackey, for whom see *Notes and Queries,* 1st S., viii. 468, 565, ix. 89, 179. Had it not been for actual quotations given by one correspondent only (1st S., viii. 565), that journal would have handed him down as a man of some real learning. An extraordinary man he certainly was: it is not one illiterate shoemaker in a thousand who could work upon such a singular mass of Sanscrit and Greek words, without showing evidence of being able to read a line in any language but his own, or to spell that correctly. He was an uneducated Godfrey Higgins. A few extracts will put this in a strong light: one for history of science, one for astronomy, and one for philology:—

"Sir Isaac Newton was of opinion that 'the atmosphere of the earth was the sensory of God; by which he was enabled to see quite round the earth': which proves that Sir Isaac had no idea that God could see through the earth."

"Sir Richard [Phillips] has given the most rational explanation of the cause of the earth's elliptical orbit that I have ever seen in print. It is because the earth presents its watery hemisphere to the sun at one time and that of solid land the other; but why has he made his Oxonian astonished at the coincidence? It is what I taught in my attic twelve years before."

"Again, admitting that the Eloim were powerful and intelligent beings that managed these things, we would accuse *them* of being the authors of all the sufferings of Chrisna. And as they and the constellation of Leo were below the horizon, and consequently cut off from the end of the zodiac, there were but eleven constellations of the zodiac to be seen; the three at the end were wanted, but those three would be accused of bringing Chrisna into the troubles which at last ended in his death. All this would be expressed in the Eastern language by saying that Chrisna was persecuted by those Judoth Ishcarioth!!!!! [the five notes of exclamation are the author's]. But the astronomy of those distant ages, when the sun was at the south pole in winter, would leave five of those Decans cut off from our view, in the latitude of twenty-eight degrees; hence Chrisna died of wounds from five Decans, but the whole five may be included in Judoth Ishcarioth! for the

phrase means *the men that are wanted at the extreme parts.* Ishcarioth is a compound of *ish*, a man, and *carat* wanted or taken away, and *oth* the plural termination, more ancient than *im*. . . . ."

I might show at length how Michael is the sun, and the D'-ev-'l, in French Di-ob-al, also 'L-evi-aith-an—the *evi* being the radical part both of d*evi*l and l*evi*athan—is the Nile, which the sun dried up for Moses to pass: a battle celebrated by Jude. Also how *Moses*, the same name as *Muses*, is from *mesha*, drawn out of the water, "and hence we called our land which is saved from the water by the name of *marsh*." But it will be of more use to collect the character of S. A. M. from such correspondents of *Notes and Queries* as have written after superficial examination. Great astronomical and philological attainments; much ability and learning; had evidently read and studied deeply; remarkable for the originality of his views upon the very abstruse subject of mythological astronomy, in which he exhibited great sagacity. Certainly his views were *original;* but their sagacity, if it be allowable to copy his own mode of etymologizing, is of an *ori-gin-ale* cast, resembling that of a person who puts to his mouth liquors both distilled and fermented.

No. XII. 1825.

John Walsh, of Cork (1786—1847).—This discoverer has had the honour of a biography from Prof. Boole, who, at my request, collected information about him on the scene of his labours. It is in the *Philosophical Magazine* for November, 1851, and will, I hope, be transferred to some biographical collection where it may find a larger class of readers. It is the best biography of a single hero of the kind that I know. Mr. Walsh introduced himself to me, as he did to many others, in the anterowlandian days of the Post-Office; his unpaid letters were double, treble, &c. They contained his pamphlets, and cost their weight in silver: all have the name of the author and all are in octavo or in quarto letter-form; most are in four pages; and all dated from Cork. I have the following by me:—

The Geometric Base. 1825.—The theory of plane angles. 1827.—Three letters to Dr. Francis Sadleir. 1838.—The invention of polar geometry. By Irelandus. 1839.—The theory of partial functions. Letter to Lord Brougham. 1839.—On the invention of polar geometry. 1839.—Letter to the Editor of the Edinburgh Review. 1840.—Irish Manufacture. A new method of tangents. 1841.—The normal diameter in curves. 1843.—Letter to Sir R. Peel. 1845. [Hints that Government should compel the introduction of Walsh's Geometry into Universities.]—Solution of Equations of the higher orders. 1845.

Besides these, there is a "Metalogia," and I know not how many others.

Mr. Boole, who has taken the moral and social features of Walsh's delusions from the commisserating point of view, which makes ridicule out of place, has been obliged to treat Walsh as Scott's Alan Fairford treated his client Peter Peebles; namely, keep the scarecrow out of court while his case was argued. My plan requires me to bring him in: and when he comes in at the door, pity and sympathy fly out at the window. Let the reader remember that he was not an ignoramus in mathematics: he might have won his spurs if he could have first served as an esquire. Though so illiterate that even in Ireland he never picked up anything more Latin than *Irelandus,* he was a very pretty mathematician spoiled in the making by intense self-opinion.

This is part of a private letter to me at the back of a page of print: I had never addressed a word to him:—

"There are no limits in mathematics, and those that assert there are, are infinite ruffians, ignorant, lying blackguards. There is no differential calculus, no Taylor's theorem, no calculus of variations, &c., in mathematics. There is no quackery whatever in mathematics; no $\frac{0}{0}$ equal to anything. What sheer ignorant blackguardism that!

"In mechanics the parallelogram of force is quackery, and is dangerous; for nothing is at rest, or in uniform, or in rectilinear motion, in the universe. Variable motion is an essential property of matter. Laplace's demonstration of the parallelogram of forces is a begging of the question; and the attempts of them all to show that the difference of twenty minutes between the sidereal and actual revolution of the earth round the sun arises from the tugging of the Sun and Moon at the pot-belly of the earth, without being sure even that the earth has a pot-belly at all, is perfect quackery. The said difference arising from and demonstrating the revolution of the Sun itself round some distant centre."

In the letter to Lord Brougham we read as follows:—

"I ask the Royal Society of London, I ask the Saxon crew of that crazy hulk, where is the dogma of their philosophic god now?....When the Royal Society of London, and the Academy of Sciences at Paris, shall have read this memorandum, how will they appear? Like two cur dogs in the paws of the noblest beast of the forest....Just as this note was going to press, a volume lately published by you was put into my hands, wherein you attempt to defend the fluxions and Principia of Newton. Man! what are you about? You come forward now with your special pleading, and fraught with national prejudice, to defend, like the philosopher Grassi, the persecutor of Galileo, principles and reasoning which, unless you are actually insane, or an ignorant quack in mathematics, you know are mathematically false. What a moral lesson this for the students of the University of London from its head! Man! demonstrate corollary 3, in this note, by the lying dogma of Newton, or turn your thoughts to something you understand. "WALSH IRLANDUS."

Mr. Walsh—honour to his memory—once had the consideration to save me postage by addressing a pamphlet under cover to a Member of Parliament, with an explanatory letter. In that letter he gives a candid opinion of himself:—

(1838.) "Mr. Walsh takes leave to send the enclosed corrected copy to Mr. Hutton as one of the Council of the University of London, and to save postage for the Professor of mathematics there. He will find in it geometry more deep and subtle, and at the same time more simple and elegant, than it was ever contemplated human genius could invent."

He then proceeds to set forth that a certain "tomfoolery lemma," with its "tomfoolery" superstructure, "never had existence outside the shallow brains of its inventor," Euclid. He then proceeds thus:—

"The same spirit that animated those philosophers who sent Galileo to the Inquisition animates all the philosophers of the present day without exception. If anything can free them from the yoke of error, it is the [Walsh] problem of double tangence. But free them it will, how deeply soever they may be sunk into mental slavery—and God knows that is deeply enough: and they bear it with an admirable grace; for none bear slavery with a better grace than tyrants. The lads must adopt my theory. ....It will be a sad reverse for all our great professors to be compelled to become schoolboys in their gray years. But the sore scratch is to be compelled, as they had before been compelled one thousand years ago, to have recourse to Ireland for instruction."

The following "Impromptu" is no doubt by Walsh himself: he was more of a poet than of an astronomer:—

"Through ages unfriended,
With sophistry blended,
Deep science in Chaos had slept;
Its limits were fettered,
Its voters unlettered,
Its students in movements but crept.
Till, despite of great foes,
Great WALSH first arose,
And with logical might did unravel
Those mazes of knowledge,
Ne'er known in a college,
Though sought for with unceasing travail.
With cheers we now hail him,
May success never fail him,
In Polar Geometrical mining;
Till his foes be as tamed
As his works are far-famed
For true philosophic refining."

Walsh's system is, that all mathematics and physics are wrong: there is hardly one proposition in Euclid which is demonstrated.

His example ought to warn all who rely on their own evidence to their own success. He was not, properly speaking, insane; he only spoke his mind more freely than many others of his class. The poor fellow died in the Cork union, during the famine. He had lived a happy life, contemplating his own perfections, like Brahma on the lotos-leaf.

No. XIII. 1825—1830.

The motion of the Sun in the Ecliptic, proved to be uniform in a circular orbit....with preliminary observations on the fallacy of the Solar System. By Bartholomew Prescott, 1825, 8vo.

The author had published, in 1803, a "*Defence of the Divine System*, which I never saw; also, *On the inverted scheme of Copernicus*. The above work is clever in its satire.

Manifesto of the Christian Evidence Society, established Nov. 12, 1824. Twenty-four plain questions to honest men.

There are two broadsides of August and November, 1826, signed by Robert Taylor, A.B., Orator of the Christian Evidence Society. This gentleman was a clergyman, and was convicted of blasphemy in 1827, for which he suffered imprisonment, and got the name of the *Devil's Chaplain*. The following are quotations:—

"For the book of Revelation, there was no original Greek at all, but *Erasmus* wrote it himself in Switzerland, in the year 1516. Bishop Marsh, vol. i., p. 320."—"Is not God the author of your reason? Can he then be the author of anything which is contrary to your reason? If reason be a sufficient guide, why should God give you any other; if it be not a sufficient guide, why has he given you *that?*"

I remember a votary of the Society being asked to substitute for *reason* "the right leg," and for *guide* "support," and to answer the two last questions: he said there must be a quibble, but he did not see what. It is pleasant to reflect that the *argumentum a carcere* is obsolete. One great defect of it was that it did not go far enough: there should have been laws against subscriptions for blasphemers, against dealing at their shops, and against rich widows marrying them.

Had I taken in theology, I must have entered books against Christianity. I mention the above, and Paine's *Age of Reason*, simply because they are the only English modern works that ever came in my way without my asking for them. The three parts of the *Age of Reason* were published in Paris 1793, Paris 1795, and New York, 1807. Carlisle's edition is of London, 1818, 8vo. It

must be republished when the time comes, to show what stuff governments and clergy were afraid of at the beginning of this century. I should never have seen the book, if it had not been prohibited; a bookseller put it under my nose with a fearful look round him; and I could do no less, in common curiosity, than buy a work which had been so complimented by church and state. And when I had read it, I said in my mind to church and state—"Confound you! you have taken me in worse than any reviewer I ever met with." I forget what I gave for the book, but I ought to have been able to claim compensation somewhere.

Cabbala Algebraica. Auctore Gul. Lud. Christmann. Stuttgard, 1827, 4to.

Eighty closely printed pages of an attempt to solve equations of every degree, which has a process called by the author *cabbala.* An anonymous correspondent spells *cabbala* as follows, χαββαλλ, and makes 666 out of its letters. This gentleman has sent me, since my budget commenced, a little heap of satirical communications, each having a 666 or two; for instance, alluding to my remarks on the spelling of *chemistry,* he finds the fated number in χιμεια. With these are challenges to explain them, and hints about the end of the world. All these letters have different fantastic seals; one of them with the legend "Keep your temper,"—another bearing "Bank token fivepence." The only signature is a triangle with a little circle in it, which I interpret to mean that the writer confesses himself to be the round man stuck in the three-cornered hole, to be explained as in Sydney Smith's joke.

The Celtic Druids. By Godfrey Higgins, Esq., of Shellow Grange, near Doncaster. London, 1827, 4to.

Anacalypsis, or an attempt to draw aside the veil of the Saitic Isis: or an inquiry into the origin of languages, nations, and religions. By Godfrey Higgins, &c.....London, 1836, 2 vols. 4to.

The first work had an additional preface and a new index in 1829. Possibly, in future time, will be found bound up with copies of the second work two sheets which Mr. Higgins circulated among his friends in 1831, the first a "Recapitulation," the second "Book vi., ch. 1."

The system of these works is that—

"The Buddhists of Upper India (of whom the Phenician Canaanite, Melchizedek, was a priest) who built the Pyramids, Stonehenge, Carnac, &c., will be shown to have founded all the ancient mythologies of the world, which, however varied and corrupted in recent times, were originally one, and that one founded on principles sublime, beautiful, and true."

These works contain an immense quantity of learning, very honestly put together. I presume the enormous number of facts, and the goodness of the index, to be the reasons why the *Anacalypsis* found a permanent place in the *old* reading-room of the British Museum, even before the change which greatly increased the number of books left free to the reader in that room.

Mr. Higgins, whom I knew well in the last six years of his life, and respected as a good, learned, and (in his own way) *pious* man, was thoroughly and completely the man of a system. He had that sort of mental connexion with his theory that made his statements of his authorities trustworthy: for, besides perfect integrity, he had no bias towards alteration of facts: he saw his system in the way the fact was presented to him by his authority, be that what it might.

I never could quite make out whether Godfrey Higgins took that system which he traced to the Buddhists to have a Divine origin or to be the result of good men's meditations. Himself a strong theist, and believer in a future state, one would suppose that he would refer a *universal* religion, spread in different forms over the whole earth from one source, directly to the universal Parent. And this I suspect he hid, whether he knew it or not. The external evidence is balanced. In his preface he says—

"I cannot help smiling when I consider that the priests have objected to admit my former book, *the Celtic Druids*, into libraries, because it was antichristian; and it has been attacked by Deists, because it was superfluously religious. The learned Deist, the Rev. R. Taylor [already mentioned] has designated me as *the religious* Mr. Higgins."

The time will come when some profound historian of literature will make himself much clearer on the point than I am.

The triumphal Chariot of Friction: or a familiar elucidation of the origin of magnetic attraction, &c. &c. By William Pope. London, 1829, 4to.

Part of this work is on a dipping-needle of the author's construction. It must have been under the impression that a book of naval magnetism was proposed, that a great many officers, the Royal Naval Club, &c., lent their names to the subscription list. How must they have been surprised to find, right opposite to the list of subscribers, the plate presenting "the three emphatic letters J. A. O." And how much more when they saw it set forth that if a square be inscribed in a circle, a circle within that, then a square again, &c., it is impossible to have more than fourteen circles, let the first circle be as large as you please. From this the seven

attributes of God are unfolded. And further, that all matter was *moral,* until Lucifer *churned* it into *physical* "as far as the third circle in Deity": this Lucifer, called Leviathan in Job, being thus the moving cause of chaos. I shall say no more, except that the friction of the air is the cause of magnetism.

Remarks on the Architecture, Sculpture, and Zodiac of Palmyra; with a Key to the Inscriptions. By B. Prescot. London, 1830, 8vo.

Mr. Prescot gives the sign of the zodiac a Hebrew origin.

Epitome de mathématiques. Par F. Jacotot, Avocat. 3ième édition. Paris, 1830, 8vo. (pp. 18).

Méthode Jacotot. Choix de propositions mathématiques. Par P. Y. de Séprés. 2nde édition. Paris, 1830, 8vo. (pp. 82).

Of Jacotot's method, which had some vogue in Paris, the principle was *Tout est dans tout,* and the process *Apprendre quelque chose, et à y rapporter tout le reste.* The first tract has a proposition in conic sections and its preliminaries: the second has twenty exercises, of which the first is finding the greatest common measure of two numbers, and the last is the motion of a point on a surface, acted on by given forces. This is topped up with the problem of sound in a tube, and a slice of Laplace's theory of the tides. All to be studied until known by heart, and all the rest will come, or at least join on easily when it comes. There is much truth in the assertion that new knowledge hooks on easily to a little of the old, thoroughly mastered. The day is coming when it will be found out that crammed erudition, got up for examinations, does not cast out any hooks for more.

Lettre à MM. les Membres de l'Académie Royale des Sciences, contenant un développement de la réfutation du système de la gravitation universelle, qui leur a été présentée le 30 août, 1830. Par Félix Passot. Paris, 1830, 8vo.

Works of this sort are less common in France than in England. In France there is only the Academy of Sciences to go to: in England there is a reading public out of the Royal Society, &c.

(*To be continued.*)

NEW WORKS.

English Life Table; Table of Lifetimes, Annuities, and Premiums. With an Introduction by William Farr, M.D., F.R.S., D.C.L.

In his letter to the Registrar-General, Dr. Farr states that the new English Life Table has been calculated from the returns of two censuses, and 6,470,720 deaths registered in 17 years, and that the tables based upon it are so constructed as to show the value of annuities on the life of a male or of a female, and on two lives—namely, two males, two females, one male and one female—at all the various combinations of age; and further, that from the given logarithms of all the combinations of $D_{x,\,y,}$ and the logatithms of l_x and l_y, the values of annuities and insurances on three lives, or on any number of lives of various ages, are readily deducible. Dr. Farr also speaks of the use of the tables in exhibiting the probabilities of English life and the values of money depending upon its duration under a great variety of circumstances, and of their use in their application to life insurance and the determination of premiums, &c. This is, as Dr. Farr no doubt intended it to be, a popular description of the vast collection cf tables contained in the volume before us. The description, however, affords but a faint idea of the infinite number and variety of values to be derived from them. Independently of a very complete set of monetary tables, those dealing with the mortality supply every conceivable quantity which the data can furnish, and in combination with interests at many rates per cent. give for either sex the several items D_x, N_n, S_x, C_x, M_x, R_x, and some of these even at half-yearly and quarterly rates of interest, the logarithmic values as well as the natural numbers being tabulated for a large portion. Dr. Farr mentions that several of the series were not only calculated but printed by the machine constructed by the Messrs. Scheutzes, and nothing can be more distinct and well defined than the figures thus produced. An interesting account of the machine, and of its mode of operation, is given in an appendix.

By way of introduction to the tables, Dr. Farr has prefixed a very elegant treatise, in four parts, which he entitles as follows:—

Part 1. Construction of the English Life Table, No. 3.
Part 2. Description of the tables.
Part 3. Analysis of values as affected by time, interest, and contingencies of life or commerce: notation.
Part 4. Synopsis of symbols, formulas and practical rules for the use of the tables.

In each of these the subject is admirably treated, and a vast deal of information conveyed in a condensed but philosophical and perspicuous way. In Part 3 some remarkable extensions of the ordinary theories are given under

the head of values at constant risk. The notation throughout appears to us to be unexceptionable, and scarcely to be improved, save in one particular—the substitution, namely, of S_x, N_x, &c., for the S_{x-1}, N_{x-1}, of Mr. Davies. We confess to a preference for the latter arrangement.

A Treatise on the Valuation of Life Contingencies, arranged for the use of Students. By EDWARD SANG, F.R.S.E.

Mr. Sang has arranged his work in a series of articles, and has divided these into two classes, the one being intended for the perusal of beginners, and the other for that of more advanced students. The articles in the latter class are distinguished by having their initial words printed in capital letters. One of the most remarkable features in this book is its entire want of similarity to any other on the same subject. The order of the investigations is *sui generis*, and so is the manner of conducting them. Still more peculiar is the notation. What will our readers say to the expression

$$\text{ann}\frac{b,\ c}{a} = \text{ann}(b,\ c) - \text{ann}(a,\ b,\ c),$$

adopted by the author for the value of an annuity during the joint lives of b and c, at decease of a? or to such forms as ann a, log liv a, log pay a, ali a, &c. Mr. Sang justifies the use of these abbreviations by the precedent afforded in trigonometrical formulæ. But we would submit that the cases are not altogether parallel; the words sine, tangent, secant, &c., are solely used technically, whilst annuity, living, payment, aliment, are words in ordinary daily use. The work is, however, a most able and original one, and whilst very useful to every student, will probably be of more than ordinary interest to minds of a certain class, for which the formidable integrals obtained by Mr. Sang for values of different descriptions will, no doubt, have a charm, impracticable as they are. A great many questions are appended as exercises, each of the articles having its own quota.

Fifth Annual Report of the Superintendent of the Insurance Department, State of New York.

These Reports are highly creditable to the ability as well as the industry of the Superintendent. Whether the powers with which he is vested are quite consistent with that freedom of action, the maintenance of which is considered of such vast importance in this country, it is not our business to inquire. Certain it is, that Mr. Barnes appears to exercise his important functions with great impartiality and great discretion, and by these qualities seems able to render a system of surveillance almost palatable, which in less scrupulous hands we should think must become intolerable. But it is with

the Report itself that we have to do, and we refer to it now mainly to call attention to some expressions of opinion from the author of it which we think deserve to be recorded here, and which the readers of this *Journal* will, no doubt, be glad to see generally adopted.

Thus, in reference to the practice of life assurance, Mr. Barnes says:—

"It is advocated on high grounds of duty, benevolence and religion, for the benefit of helpless infancy, disconsolate widowhood and unprotected orphanage; its official expositors are, therefore, bound to exhibit in their corporate affairs a like spirit and exalted aspiration, and to be governed by a standard of morality rigidly exacting and scrupulously honourable, above and beyond even any legal requirement or interdiction of statute law. The glowing promises of the prospectus should remain as sacred pledges, never to be violated. Low chicane, vulgar trickery, loose oaths and evasive statements, should all be banished from the armoury of the life underwriter. The severest rules of the strictest justice, and the keenest honour, both in guarding and dispensing their funds, should naturally become incarnated in the life manager. Officers incapacitated by nature or inclination from assuming such relations, have mistaken their true vocation; the cold-hearted speculator and leech-like stockholder should return to their appropriate fields of danger and adventure, and leave the trust funds of other generations to the sleepless care and watchful vigilance of those rare men seemingly born for such sacred guardianship.

"In requiring such an elevated standard for the regulation of Life Insurance Corporations, the superintendent is influenced by a strong conviction of the high and responsible functions which our advancing civilization and politico-economic tendencies will devolve upon their managing officers. With the modifications and improvements in our present system, which a few years more will naturally develop and test by actual results, the practice of life insurance must become vastly increased and almost universal. The evils resulting from lapsed policies should, if possible, be in some manner ameliorated, and more liberal rules be adopted as to change of residence and travelling. Indeed, 'whole-world' policies harmonize more with American ideas than extra premiums, and any system of permits, however liberal."

On the subject of extravagant commission, he observes:—

"It is not to be denied, that in the fervid and zealous competition indulged in by the Companies, some Offices have been tempted to over-step the bounds of prudence and propriety in regard to the amount of commissions and other compensation paid to successful agents and canvassers. Fifty, forty, thirty, or even twenty-five per cent. commissions, on first premiums, are extravagant largesses or bribes for the procuration of policies, however skilfully the process may be disguised by an attempt theoretically to spread such payments over future years; and the Superintendent does not hesitate to denounce this false and fatal system as destructive to the Companies and detrimental to the public, and as an inexcusable violation of the plain duties of officers in the performance of their official and public functions, even when premiums are so calculated and loaded as to allow of such a margin of expenditure. The over-payment or surplus of premiums

constitutes a trust fund, to be as sacredly guarded as the accumulations resulting from the pure premium."

And lastly, as to the managers and actuaries of Life Assurance Companies, Mr. Barnes says:—

"The business of life insurance is liable to greater peril from the ill-management of Companies, than from any probable increase in their number. The greatest care and circumspection should be used in the selection of all managing and actuarial officers of a Life Company. Changes are seldom made, and when made are attended with many dangers and difficulties. As a general rule such officers have been selected with admirable skill and judgment, and no class of Corporations in the Union are, in the main, under the guardianship of more able, honest, and reliable men. Still, when the morality of the purest minister of the Gospel, the knowledge and ability of the most learned jurist, and the tact and skill of the ablest financier, coupled with the highest mathematical and medical acquirements, and with rare executive and administrative abilities, are all needed, combined either in one or in several managers, it is not to be denied that some of these attributes are still outside of the active direction of a portion of our Life Corporations. The acquisition, even at considerable expense, of these talents, is a most urgent necessity demanding the immediate attention of such institutions."

There are other passages in the Report well worthy of the English reader's perusal, but we must refer those who have a desire to read them to the document itself.

Tables for comparing British with Metric Measures and Weights. By CHARLES HUTTON DOWLING. Lockwood & Co.

Weights and Measures of all Nations, and an Analysis of the Christian, Hebrew, and Mahometan Calendars. By W. S. B. WOOLHOUSE, F.R.A.S., &c. Virtue Brothers & Co.

Archiv für das Versicherungswesen Heransgegeben in Zwanglossen-Heften. Von DR. A. F. ELSNER, Redacteur der Deuts-chen Versicherungs-Zeitung. Berlin.

We hope to give some account of these productions in a future Number.

CORRESPONDENCE.

ON MR. BAILEY'S ESTIMATE OF THE LIABILITIES OF CERTAIN LIFE ASSURANCE COMPANIES.

To the Editor of the Assurance Magazine.

SIR,—It is not usual to discuss, in the *Assurance Magazine*, the accounts of individual Companies, or the merits of the various methods of division of profits employed by them; and it would be matter of regret if such discussions were generally allowed to appear in your pages. But when there is reason to think that an injustice has been done to a particular Company by one of your correspondents, I doubt not that you will allow your usual salutary rule to be relaxed so far as may be necessary to correct that injustice.

I refer to Mr. Bailey's table, which appeared in the *Assurance Magazine* for July, 1863 (vol. xi., pp. 111, 112), and purports to show the "estimated liability" of various Offices. I may say for myself that I entertain a very low opinion of the value of such general comparisons as are afforded by the table in question. Mr. Bailey has pointed out one disturbing element in the numerous recent amalgamations which have occurred; but this is far from being the only cause which interferes with the fairness of the comparison, or the most important one. It is obvious that if, of two Offices of the same standing, one has done a new business generally increasing in amount from year to year, while the new business of the other has been rather falling off, then the liability of the latter must be greater in proportion to the sum assured than is the case with the former. There is at least one marked instance of the effect of this cause apparent in Mr. Bailey's table (p. 112).

In order that such tables may have any value at all, it is of course essential that the figures given in them should be accurate, and that they should be deduced from the published statements of the Offices by a process which admits of no dispute. Now, there are two Offices, which I will call (A) and (B), included in Mr. Bailey's list, which appear to make a much smaller reserve for their liabilities than most other Offices of similar standing; but this is not really the case, as I will presently show. In fact, it appears to me that the accounts of these two Offices have been treated erroneously, and that the correct sums to be tabulated as the values of the estimated liability are much larger than those given in the table. In the Office A, which had been established for 54* years on 30th June, 1862, the sum assured at that date was £6,526,853; and the "estimated liability" is given by Mr. Bailey as only £775,002. Relying on the accuracy of these figures, I have on various occasions expressed myself, in the course of conversation, to the effect that "the Office in question makes a very small reserve for its liabilities;" and it cannot be disputed that this is the obvious conclusion to be drawn from the table. Having made the above statement to a friend who is insured in it, he challenged me to prove my assertion, and placed in my hands the printed accounts of the Association for a series of years; and I have now to acknowledge that an examination of these accounts has led me to a conclusion very different from Mr. Bailey's.

* More accurately 55½.—ED. *A. M.*

In order to ascertain the reserve actually made by the Office (A) for its liabilities, it will be convenient to give a summary of the balance sheet of the Association on 30th June, 1862. This is as follows:—

Liabilities.

Value of £4,758,993 assured on the lives of members—1st series	£ 2,716,457
Value of £1,523,570 assured on the lives of members—2nd series	596,445
Value of £244,290 assured on the lives of persons not members	110,177
Sundry liabilities	102,444
	£3,525,523

Assets.

Sundry assets	2,830,539
Value of £5,612 premiums on non-members' policies	59,349
Value of such part of the annual premiums, amounting to £56,193, on the lives of members (2nd series), as they will be required to pay in full	171,355
Value of the future reduced premiums on the lives of members:—	
1st series, at $83\frac{1}{2}$ per cent. reduction	290,934
2nd " $73\frac{1}{2}$ "	173,346
	£3,525,523

The first thing to be observed in this account is, that while the sum assured is, as stated by Mr. Bailey, £6,526,853, the realized assets, after allowing for all immediate liabilities, amount to £2,728,095, or no less than 41·8 per cent. of the sum assured. How, then, is Mr. Bailey's result obtained? It will be found that the amount tabulated by him (£775,002) is the difference between the value of the sums assured (£3,423,079) and the value of the future gross premiums (£2,648,077), the latter amount being deduced from the figures in the balance sheet by a few simple calculations which it is not necessary to describe. Or Mr. Bailey's result may be obtained in the following way:—The value of the future reduction of premium—at $83\frac{1}{2}$ per cent. on the 1st series and $73\frac{1}{2}$ per cent. on the second series—is found to be £1,953,093; and the difference between this amount and the net assets (£2,728,095) is £775,002—the amount tabulated by Mr. Bailey. The relation between these figures will be more clearly seen from the following statement:—

Liabilities.

Value of sums assured	£3,423,079
" reduction of premium	1,953,093
	£5,376,172

Assets.

Value of gross premiums	£2,648,077
Net realized assets	2,728,095
	£5,376,172

It thus appears that Mr. Bailey has deducted from the assurance fund of the Association the value of the future reduction of premium; and I presume he would justify this course by arguing, that although the reduction of premium is variable and uncertain, being declared from year to year, yet the hope is held out by the Association that the present reduction may be permanently maintained. Assuming for the present that the reduction is to be considered permanent in the case of (A) and (B), then I have to observe that the accounts of these Offices are treated in Mr. Bailey's table in a very different manner to those of the other Offices mentioned. In all the other instances the values of the bonuses declared, and of any permanent reductions of premium, are included in the "estimated liability"; and in one instance—the Equitable—it is pointed out that the magnitude of the bonuses has a great influence in raising the amount of the estimated liability. Consistency, therefore, certainly requires that if the reduction of premium in (A) is considered as a permanent thing, its value ought to be included in the estimated liability, which is therefore raised to £2,728,095; amounting, as already stated, to 41·8 per cent. on the sum assured.

But there can be no doubt that the above is not the correct way of regarding the reductions of premium declared by the two Offices in question. Those reductions are fixed every year in conformity with the results of the valuation then made; and are subject to increase or diminution, from year to year, according as the experience of the Office in the past year has been favourable, or the contrary. It is therefore more correct, as well as more convenient for the purpose of comparison with other Offices, to say that an annual cash bonus is declared, equal to the value of the abatement of premium allowed for the ensuing year. Thus then the directors of (A), in submitting to the members the balance sheet of which a summary has been given above, are to be regarded as saying—"We declare an abatement of premium for the year ending 30th June, 1863, of 83½ per cent. for the members of the 1st series, and 73½ per cent. for such members of the second series as are entitled to an abatement; and these abatements are calculated on such a scale that they may reasonably be expected to be maintained from year to year so long as the experience of the Society continues similar to its present experience." An examination of the balance sheet leads to the conclusion that the abatements may be maintained on the present scale, so long as the expenses of management are defrayed out of the excess of interest realized over the rate at which the valuations are made (which is currently reported to be 4 per cent.); and the incidental profits arising from surrenders, from the premiums upon non-members' policies, &c.;—provided the mortality experienced does not exceed that calculated upon in the table of mortality used in the valuations.

In order to reconstruct the balance sheet in a proper form for comparison with the accounts of other Societies, it is necessary to ascertain approximately the amount of the abatement of premium for the year

ending 30th June, 1863. It is stated in the printed account of the Association from which the above figures are taken, that the gross annual premiums on all existing policies amount to £228,800; that the premiums on non-members' policies are £5,612, and the premiums on the policies of members of the 2nd series are £56,193. It follows, therefore, that the annual premiums on the policies of members of the 1st series are £166,995, and the abatement at $83\frac{1}{2}$ per cent. amounts to £139,441. In the particular year under consideration it will be found that all the members of the 1st series were entitled to abatement, having paid seven premiums, but none of the second series were yet entitled to an abatement. Now, arranging the balance sheet in the form usually adopted when the value of the gross premiums is stated, it will be as follows:—

Liabilities.

Value of sums assured	£3,423,079
Reserve for expenses and future bonuses .	1,813,652
Balance, being divisible surplus . .	139,441
	£5,376,172

Assets.

Investments	£2,728,095
Value of future premiums . . .	2,648,077
	£5,376,172

It will be noticed, however, that in strict accuracy the surplus should be the value of the year's abatement of premium, instead of the amount of that abatement. It thus appears that when the accounts of this Office are treated in the same way as those of the other Societies considered by Mr. Bailey, it is found that, on 30th June, 1862, no less than $68\frac{1}{2}$ per cent. of the value of the future gross premiums was reserved for expenses and future bonuses, and the "estimated liability" was (£2,728,095 — £139,441), or £2,588,654, instead of £775,002, as given by Mr. Bailey; or the estimated liability is 39·7 per cent. on the sum assured, instead of 11·9.

Similar remarks apply to the Office (B), which had been 28 years in existence on 4th April, 1863, and which is represented as making a reserve equal to 9·4 per cent. of the sum assured. In this case, the sum assured is £3,375,224, and the available assets, after making provision for immediate liabilities, are £964,275, or 28·6 of the sum assured. I gather from the published accounts of this Office, that the amount of the abatement allowed to members in the year 1863–4 was £43,226; and deducting this sum from the available assets, as above, it appears that the "reserve" of the Office is £921,049, or 27·3 on the sum assured, instead of 9·4.

There can thus, I think, be no doubt that the reserves made by (A) and (B) for their liabilities, are not unduly small as compared with the reserves of other Offices; but whether the method employed to ascertain the abatement of premium for each year, is the most suitable, is a totally distinct question, upon which it would not be proper to enter here.

That the method of treating the accounts adopted in the table is erroneous will perhaps be rendered more clear to some persons by the consideration of

the case of an Office which should make an annual valuation and declare a reversionary bonus thereupon each year. Such an Office might declare, for a series of years, a reversionary bonus at the rate of $1\frac{1}{2}$ per cent. per annum on the sum assured, and the expectation might be held out to the assured that this rate of bonus would be maintained. In this case it would be manifestly improper to deduct from the assurance fund of the Society the value of the future reversionary bonus at the above rate, and style the balance the "estimated liability" of the Office. But this is precisely analogous to what has been done with the Offices (A) and (B).

I have already trespassed much on your space, but as I believe that a general interest is felt in our profession on these points, I will proceed to the consideration of another method by which the sufficiency and the magnitude of the reserve made by the Offices in question may be tested—viz., the valuation of individual policies. Take the case of a policy for £100 effected in the Office (A) at the age 40, at the annual premium £3. 17*s.*, which has been in force m years ($m<7$); then the reserve made for this policy by the method of valuation pursued, is

$$100A_{40+m}-3{\cdot}85\times(1+{}_{\overline{6-m}|}a_{40+m})-1{\cdot}02a_{40+m\overline{6-m}|}$$

$$=100A_{40+m}-3{\cdot}85(1+a_{40+m})+2{\cdot}83a_{40+m\overline{6-m}|}$$

$$i.e.,\ V_{40|m}=100A_{40+m}-3{\cdot}85(1+a_{40+m})+2{\cdot}83\frac{N_{46}}{D_{40+m}}\quad .\ .\ .\ .\quad (1),$$

For the premium £3·85 is payable in advance for $7-m$ years, and afterwards the premium is supposed to be reduced 73·5 per cent., or to become £1·02.

Again, for a similar policy on the life of a member of the first series, which has been in force 7 years or more (say $7+n$ years), and upon which, therefore, there is a reduction of 83·5 per cent. on the premiums, the reduction being £3·21475, and the reduced premium £·63524, the value of the policy is

$$V_{40|7+n}=100A_{47+n}-{\cdot}63525(1+a_{47+n})\quad .\ .\ .\ .\quad (2).$$

By means of the formulæ (1) and (2), the values in columns (2) and (5) of the following table have been calculated (at Carlisle 4 per cent.).

Table of Values of Policies.

Policies on the Lives of Members of 2nd Series.			Policies on the Lives of Members of 1st Series.		
	Formula (1).	Carlisle 3 per Cent.		Formula (2).	Carlisle 3 per Cent.
$V_{40\|1}$	7·790	1·394	$V_{40\|7}$	27·863	10·191
$V_{40\|2}$	10·882	2·773	$V_{40\|8}$	28·953	11·889
$V_{40\|3}$	14·089	4·155	$V_{40\|9}$	30·143	13·718
$V_{40\|4}$	17·451	5·585	$V_{40\|10}$	31·418	15·653
$V_{40\|5}$	20·983	7·056	$V_{40\|11}$	32·776	17·698
$V_{40\|6}$	24·713	8·589	$V_{40\|12}$	34·155	19·760
			$V_{40\|20}$	52·243	36·664
			$V_{40\|30}$	65·452	55·228
			$V_{40\|40}$	76·772	70·429

From a comparison of these values, two important conclusions may be drawn. Firstly, it should seem that the reserve made by the Office is greatly in excess, in the instances here given, of the values of the policies as given by the Carlisle 3 per cent. table; and it will be found, I believe, throughout the whole of life, that the reserve made by the valuation of the Office is in excess of that required by the Carlisle 3 per cent. valuation; from which it results that if a valuation of the Society were made by the Carlisle 3 per cent. table, there would be a much larger cash bonus divided than is now allowed in reduction of premium. Secondly, it will be noticed that the reserve made for recent policies is for several years greater than the amount of the premiums received, so that in fact every new policy issued causes loss on the subsequent valuations—reduces the divisible surplus—and makes the abatement of the premium less than it would otherwise be.

This last observation opens up a wide and tempting field of investigation, but one which cannot be considered suitable for these pages. I therefore abstain from proceeding any further in that direction.

It will, of course, be understood that the values in the preceding table are not to be taken as the actual amounts reserved by the valuation of the Company. I believe that valuation is not conducted by the Carlisle table; and without being in possession of the table of mortality by which the valuations are conducted, it is impossible to assign the actual values of the policies. If the table in use is one which gives throughout a greater expectation of life than the Carlisle, then the values of the policies will be less than those given above; but it cannot be supposed that any table of mortality whatever would give such results as to vitiate the conclusions I have ventured to draw from a comparison of the values in the above table.

In conclusion, I should wish to add, that in writing these remarks I have not been in any way actuated by a desire to recommend the system pursued by the two Offices I have alluded to. I do not feel at liberty to express in these pages any opinion as to the merits of that system; and in all that I have said I have been careful to abstain from any expression of opinion, and to confine myself strictly to the discussion of questions of fact.

I am, Sir,

Your obedient servant,

T. B. SPRAGUE.

Equity and Law Life Assurance Society,
18, *Lincoln's Inn Fields,*
August, 1864.

ON MR. HODGE'S REMARKS UPON THREE-LIFE SURVIVORSHIPS.

To the Editor of the Assurance Magazine.

SIR,—I must beg the favour of a small space in your columns for a word or two in reference to Mr. Hodge's comments, at the last meeting of the Institute, upon Mr. Gray's account of my "Solutions of survivorship problems."

Mr. Hodge informed us that it was at one time his practice to calculate

his three-life cases by Milne's formulæ, but that he afterwards found he could attain his results with sufficient accuracy by means of Simpson's well-known rule of substituting for two joint lives an equivalent single life. That is, I suppose, in calculating the values of the survivorship reversions, $\underset{1}{\mathfrak{A}}\mathbb{B}\mathbb{C}$, $\underset{1}{\mathfrak{B}}\cdot\underset{1}{\mathfrak{C}}\mathbb{A}$, &c., Mr. Hodge *now* (like everybody else, I presume) first finds the single life D, equivalent to the joint lives BC, and then determines the values of $\underset{1}{\mathfrak{A}}\mathbb{D}$, $\underset{1}{\mathfrak{D}}\mathbb{A}$, &c.

But what are we to understand from Mr. Hodge's account of his *former* practice? Does he mean that he calculated *accurately* each of the seven three-life annuities involved in the following formula, for instance, which is one of those given by Milne, and by far the least laborious of the series:—

$$\underset{1}{\mathfrak{A}}\mathbb{B}\mathbb{C}=\frac{1}{3}\left\{v-(1-v)\mathrm{ABC}\right\}+\frac{1}{6{}_1a_1}\left(2\mathrm{A_1BC}+\frac{\mathrm{A_1B_1C}}{{}_1b_1}+\frac{\mathrm{A_1BC_1}}{{}_1c_1}\right)$$
$$-\frac{1}{6{}_1b_1}\left(\mathrm{AB_1C}+\frac{2\mathrm{AB_1C_1}}{{}_1c_1}\right)-\frac{\mathrm{ABC_1}}{6{}_1c_1},$$

or did he calculate these annuities *approximately* by Simpson's rule, and therewith determine the value of the reversion? If the former, it is to be hoped that these cases were not of frequent occurrence with him; and if the second supposition is the correct one, I think the process can scarcely afford a satisfactory test of the accuracy of other methods of solving the problem.

Again, Mr. Hodge did not explain whether his statement was confined to the *simple* survivorship problems, like the above, and others derived from them; or whether he included *also* those marked with an asterisk in in the synoptical table on page 195 of Milne's work. If he referred to the former *only*, it is scarcely necessary to point out that those problems were *not* the subject of discussion on the occasion referred to. If, on the other hand, his remarks applied to the *latter* problems *also* (which *did* form the subject of Mr. Gray's observations), Mr. Hodge is no doubt aware that Milne's solutions of these cases are but rude approximations; and, consequently, that the fact of any shorter methods giving results in near accordance with Milne's, is no proof that such results can be relied upon. Under either supposition, therefore, it is difficult to see what bearing Mr. H.'s remarks had upon the question before the meeting.

Before concluding, I may refer to one more point in Mr. Hodge's observations. In my former paper on the same subject (see *Assurance Magazine*, vol. x., p. 243), it will be seen that in the process of transforming

$$\underset{\text{III}}{\mathfrak{A}}\underset{\text{II}}{\mathbb{B}}\mathbb{C}=\underset{\text{II}}{\mathfrak{A}}\mathbb{B}-\underset{1}{\mathfrak{B}}\mathbb{A}\mathbb{C}+{}_{\rceil\underset{1}{\mathrm{BC}}}\mathrm{A}(1-v)$$

$$\text{into }\underset{\text{III}}{\mathfrak{A}}\underset{\text{II}}{\mathbb{B}}\mathbb{C}=\underset{1}{\mathfrak{A}}\cdot\underset{1}{\mathfrak{C}}\mathbb{B}-\underset{1}{\mathfrak{A}}\mathbb{B}-{}_{\rceil\underset{1}{\mathrm{BC}}}\mathrm{A}(1-v).$$

I make use, incidentally, of the identity $(\mathrm{A}-\mathrm{ABC})(1-v)=\mathfrak{ABC}-\mathfrak{A}$, making, however, no observation whatever upon it. Mr. Gray, who, as we all know, is rather curious in such matters, referred to it, in passing, as

noticeable—his remark evidently having reference, not to any supposed difficulty in obtaining the equation (for it is so simple that I use it as a self-evident proposition), but solely to the symmetry exhibited in it.

I remain, Sir,

Your very obedient servant,

London, 1*st December*, 1864. W. M. MAKEHAM.

P.S.—As stated above, Milne's formula for $\underset{1}{\mathfrak{A}}\mathbb{B}\mathbb{C}$ is much less laborious than others of the series. The following is from his 23rd problem:—

$$\underset{\text{II.III}}{\mathfrak{A}}\mathbb{B}\underset{1}{\overset{\circ}{\mathbb{C}}}=\begin{cases}\tfrac{1}{2}\mathfrak{A}-\frac{1}{6}\mathfrak{ABC}-\frac{1}{6\,{}_1a_1}\left(A_1BC-\frac{3(AB)_1-2A_1B_1C}{{}_1b_1}+\frac{3(AC)_1-A_1BC_1}{{}_1c_1}\right)-\\ \frac{1}{6\,{}_1b_1}\left(3AB_1-2AB_1C-\frac{AB_1C_1}{{}_1c_1}\right)+\frac{3AC_1-ABC_1}{6\,{}_1c_1}+(\tfrac{1}{2}-\underset{1}{b}c)_{\rceil\gamma}\mathfrak{A}\\ -\frac{1}{2_1(ab)_1}{}_{\rceil\gamma}(AB)_1+\frac{1}{2\,{}_1b_1}{}_{\rceil\gamma}AB_1\end{cases}$$

This is the solution for the case where *C* is the oldest life, for Milne's formulæ for this problem vary according to the seniority of the lives involved; and it will be borne in mind that it is at best but a rough approximation. For the above I propose to substitute the formula

$$\underset{\text{II.III}}{\mathfrak{A}}\mathbb{B}\underset{1}{\mathbb{C}}=\underset{1}{\mathfrak{C}}\mathbb{A}\mathbb{B}-{}_{\rceil BC}A(1-v),$$

which is rigidly accurate—independent of seniority—and the two terms of which, as I shall hereafter show, admit of an easy and expeditious mode of calculation.

W. M. M.

THE

ASSURANCE MAGAZINE,

AND

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On a Table for the Formation of Logarithms and Anti-Logarithms to Twelve Places. *By* PETER GRAY, F.R.A.S., *Honorary Member of the Institute of Actuaries.* (*Part II.*)

[Read before the Institute, 30th January, 1865.]

IN the last number of the *Journal* I explained and exemplified a method for the formation of logarithms and anti-logarithms to twelve places, and gave the tables requisite for its application. I am now to analyse and further exemplify the method in question, and demonstrate the rules laid down in the previous paper.

Let N be any given number, whose logarithm to t places is required.

Divide or multiply N by a, any number that will give a result having unit for its first figure. Place the decimal point after this first figure, and denote the result so modified by $1+N^1$, in which N^1 consequently will be a decimal fraction. Then, denoting by m the number of places through which the decimal point was moved (*plus* or *minus*, according as the movement was to the left or to the right), we should have

$$N=10^{\pm m}.a^{\pm 1}(1+N^1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Let $1+n_1$ denote *so many* (it is no matter for the present purpose *how* many) of the leading figures of $1+N^1$; subtract $1+n_1$ from $1+N^1$, and let the remainder be r_1; divide r_1 by $1+n_1$, and let the

quotient and the remainder be n_2 and r_2 respectively; divide r_2 by $(1+n_1)(1+n_2)$, and let the quotient and the remainder of this division be n_3 and r_3 respectively. Proceeding thus, we find that division of r_{p-1} by $(1+n_1)(1+n_2)\ldots(1+n_{p-1})$ gives for quotient and remainder n_p and r_p respectively.

This operation may be typically represented as follows :—

$$
\begin{array}{rcl}
 & 1+N^1 & \\
 & \underline{1+n_1} & \\
1+n_1\,) & r_1 & (\,n_2 \\
 & \underline{(1+n_1)n_2} & \\
(1+n_1)(1+n_2)\,) & r_2 & (\,n_3 \\
 & \underline{(1+n_1)(1+n_2)n_3} & \\
 & r_3 & \\
 & \vdots & \\
(1+n_1)(1+n_2)\ldots(1+n_{p-1})\,) & r_{p-1} & (\,n_p \\
 & \underline{(1+n_1)(1+n_2)\ldots(1+n_{p-1})n_p} & \\
 & r_p &
\end{array}
$$

From this we obviously have,

$$
\begin{aligned}
r_1 &= (1+N^1)-(1+n_1)\\
r_2 &= r_1 -(1+n_1)n_2\\
r_3 &= r_2 -(1+n_1)(1+n_2)n_3\\
&\vdots\\
r_{p-1} &= r_{p-2}-(1+n_1)(1+n_2)\ldots(1+n_{p-2})n_{p-1}\\
r_p &= r_{p-1}-(1+n_1)(1+n_2)\ldots(1+n_{p-1})n_p;
\end{aligned}
$$

or, by transposition,

$$
\begin{aligned}
(1+N^1)-r_1 &= (1+n_1)\\
r_1 - r_2 &= (1+n_1)n_2\\
r_2 - r_3 &= (1+n_1)(1+n_2)n_3\\
&\vdots\\
r_{p-2}-r_{p-1} &= (1+n_1)(1+n_2)\ldots(1+n_{p-2})n_{p-1}\\
r_{p-1}-r_p &= (1+n_1)(1+n_2)\ldots(1+n_{p-1})n_p.
\end{aligned}
$$

Now, the foregoing equations may be written as follows :—

$$
\begin{aligned}
(1+N^1)-r_1 &= (1+n_1)\\
r_1 - r_2 &= (1+n_1)(1+n_2)-(1+n_1)\\
r_2 - r_3 &= (1+n_1)(1+n_2)(1+n_3)-(1+n_1)(1+n_2)\\
&\vdots\\
r_{p-2}-r_{p-1} &= (1+n_1)(1+n_2)\ldots(1+n_{p-1})-(1+n_1)(1+n_2)\ldots(1+n_{p-2})\\
r_{p-1}-r_p &= (1+n_1)(1+n_2)\ldots(1+n_p)-(1+n_1)(1+n_2)\ldots(1+n_{p-1}).
\end{aligned}
$$

Adding together these last equations, we obtain,

$$\left.\begin{array}{l}(1+N^1)-r_p=(1+n_1)(1+n_2)(1+n_3)\ldots(1+n_p)\\ \text{or,}\quad 1+N^1=(1+n_1)(1+n_2)(1+n_3)\ldots(1+n_p)+r_p\end{array}\right\}\quad .\ (2).$$

From this it appears that, since the remainders r_2, r_3, &c., successively decrease, the product $(1+n_1)(1+n_2)\ldots$ constantly approximates to $1+N^1$; so that, if the operation above described and typified shall have been continued till r_p has no significant figures in the first t decimal places, the equation

$$1+N^1=(1+n_1)(1+n_2)(1+n_3)\ldots(1+n_p)\quad .\ .\ .\quad (3)$$

will be true to t decimal places, or to $t+1$ places in all.

We may now substitute this value of $1+N^1$ in equation (1), which thus becomes

$$N=10^{\pm m}.\,a^{\pm 1}(1+n_1)(1+n_2)\ldots(1+n_p)\quad .\ .\ .\ .\quad (4).$$

Now, we know that the first t places of a logarithm are not affected by any figure beyond the $(t+1)$th place of the corresponding number;* hence, equation (4) being true to $t+1$ places, if we take the logarithms of both sides, the resulting equation will be true to t places. So we should have

$$\log N=\pm m\pm\log a+\log(1+n_1)+\log(1+n_2)\ldots+\log(1+n_p)\ .\ (5).$$

This expression admits of certain modifications, by which the facility of its application is increased.

First. By inspection of any given number we can at once assign the index of its logarithm. We may, therefore, neglect all indices in the course of the operation; and hence m, which, being an integer, affects only the index of the logarithm, may be omitted from the formula.

Secondly. If for $-\log a$, when it occurs, we write $\operatorname{colog} a$, the component logarithms will then always have to be combined by addition.

We thus obtain for the determination of the mantissa of $\log N$, the two following formulæ :—

$$\log a+\log(1+n_1)+\log(1+n_2)\ldots+\log(1+n_p),$$

and

$$\operatorname{colog} a+\log(1+n_1)+\log(1+n_2)\ldots+\log(1+n_p);$$

* This can easily be shown by aid of the expression

$$\log(x+h)-\log x=M\left\{\frac{h}{x}-\frac{h^2}{2x^2}+\ldots\right\},$$

where M is the modulus, $=\cdot 43429\ldots$. And it is for the reason above stated that the decimal places in $1+N^1$ are, in the resolving process, restricted to t, the number of places there are to be in the mantissa of the required logarithm.

the first for use when a has been employed as a *divisor*, and the second when it has been employed as a *multiplier*.

The operation which has now been symbolically described and illustrated will, perhaps, hardly as yet have been recognised as identical with that which was employed in the previous paper for the formation of logarithms. The arithmetical illustrations I am now about to give will, however, serve to establish this identity.

For the application of these formulæ we must be provided with, first, a table of $\log a$ and $\operatorname{colog} a$; which, as a needs never be a number consisting of more than a single digit, will be simply a table of the logarithms and co-logarithms of the numbers 2 to 9. And secondly, we require a table of $\log(1+n)$ for all the values of n that can arise in the course of the operation. The extent of this table will depend mainly on the number of figures in n_1, n_2, &c.; that is, on the extent to which we carry the several divisions in the resolving process. By the terms of the preceding investigation we are laid under no restrictions in this respect, but are left at liberty to adopt the course which may seem most expedient. It is soon found that every increase in the number of figures in n_1, n_2, &c., besides possessing other advantages, materially simplifies the resolving process, while at the same time the extent of the table of $\log(1+n)$ is largely increased. A compromise, therefore, becomes necessary, which shall allow us to enjoy the greatest amount of facility in the process that may be consistent with a moderate extent of the table. I believe that, for twelve-figure logarithms, the course I have adopted is the best. I carry the divisions to three figures, and the table consists of 3,000 values, which admit of very lucid arrangement in ten pages. Had I gone a step further, and used four-figure quotients, a table consisting of 30,000 values would have been necessary.

In the following example, illustrative of the resolving process, I shall give one, two, and three-figure solutions. We shall thus be more naturally conducted from the form of the process typified above to the somewhat modified form of it employed in the previous paper. The advantage of the three-figure process over the others will thus also be rendered more apparent.

Resolve $\pi = 3{\cdot}14159{,}26535{,}90$, into factors.

The operation by the one-figure process is as follows :—

	3141592653590	÷2
1·5	~~1~~·~~5~~70796326795	4
60	60	
1·560	)10796	6
9360	9360	
1·569360	)14363267	9
141242	14124240	
1·5707724~~2~~	)23902795	1
1571	15707724	
1·570788~~1~~	)8195071	5
79	7853941	
1·57079~~6~~	)341130	2
	314159	
	26971	1
	15708	
	11263	7
	10996	
	267	1
	157	
	110	7
	110	
	0	0

	3141592653590	× 4
1·2	~~1~~·~~2~~56637061436	4
	48	
1·248	)8637	6
	7488	
1·255488	)11490614	9
	11299392	
1·2566179~~4~~	)19122236	1
	12566179	
1·256630~~6~~	)6556057	5
	6283153	
1·25663~~7~~	)272904	2
	251327	
	21577	1
	12566	
	9011	7
	8796	
	215	1
	126	
	89	7
	88	
	1	1

Two solutions are given, on which I shall remark in order. Comparing the process on the left with the type on page 122, it is seen that 2 is taken for a, and division gives for $1+N^1$, 1·5707 . . . The leading unit and the first decimal place of this, namely, 1·5, are taken for $1+n_1$. Instead of formally subtracting this from $1+N^1$, as in the type, the figures are struck out, and re-written a little to the left. Division of $r_1=\cdot07079\ldots.$ by $1+n_1$, gives $\cdot04=n_2$. We now form a new divisor. This is $(1+n_1)(1+n_2)$, and it is formed by adding to $(1+n_1)$ its product by n_2, which product is got from the adjoining column. The divisor having thus received an accession of two figures, we must, to restore the relation between divisor and remainder, annex two figures to the remainder of the previous division; and taking down another figure, we obtain by division another quotient, $\cdot006=n_3$. Forming a new divisor as before, by adding to the last the product opposite in the adjoining column, we find it contains three figures more than the preceding;

and therefore three figures have to be taken down to the remainder to restore the relation, and one figure more to allow the division to go on. This gives us $n_4 = \cdot 0009$. At this point we see that there remain only two figures to be taken down from the dividend, and it is therefore unnecessary to extend the divisor more than two places. Hence, only so much of the addend for the new divisor as will give an accession of two places is taken from the adjoining column, and the last two figures of the dividend being annexed to the remainder, the divisor is curtailed of its last figure to allow the division to proceed. This gives us $n_5 = \cdot 00001$. The next two divisors are in like manner each curtailed of a figure; and when the last of them (the sixth) is reached, it is seen that it is unnecessary to form any more, as from that point the effective figures would undergo no change. The operation, therefore, passes into that of contracted division.

The above process may be described as one of division by a variable divisor; the object and the effect of the arrangement for restoring the relation between divisor and remainder after each accession to the former being, to preserve the position in the scale which belongs to them of the several quotient figures, which is that of orderly succession from the decimal point. Writing them thus, commencing with n_1,

$$\cdot 546915217170,$$

each occupies its proper position with respect to the decimal point.

It will be observed that the sixth divisor agrees, so far as it is formed, with the figures of $1+N^1$; and if the formation were extended as far as twelve decimal places, and the succeeding products in the adjoining column added to it, the sum would be found equal to $1+N^1$. This is in accordance with equation (3).

The divisors admit of being formed in a different manner, by which the writing of a number of figures is saved.

By equation (2),

$$(1+n_1)(1+n_2)\dots(1+n_p)=(1+N^1)-r_p.$$

Now the left-hand member of this equation is, by the type, the pth divisor, and r_p is the remainder at the same point. Hence, each new divisor may be formed by subtracting the remainder, at the point of the operation attained, from the figures of $1+N^1$ directly over it. Thus, the three subtractions following give the second, third, and fourth divisors:—

$$\begin{array}{r}1\cdot570\\10\\\hline 1\cdot560\end{array}\qquad\begin{array}{r}1\cdot570796\\1436\\\hline 1\cdot569360\end{array}\qquad\begin{array}{r}1\cdot5707963267\\239027\\\hline 1\cdot5707724240\end{array}$$

These subtractions can be performed by inspection, and the results at once set down in their places.

In the resolution on the right, in which 4, used as a multiplier, is taken for a, the last method of forming the divisors is employed. Its advantage over the other will be more manifest in the case of the two and three-figure solutions.

The two sets of factors into which, in accordance with (4), π is resolved by the preceding processes, are—

$2\times 1{\cdot}5$	and $\frac{1}{4}\times 1{\cdot}2$
$\times 1{\cdot}04$	$\times 1{\cdot}04$
$\times 1{\cdot}006$	$\times 1{\cdot}006$
$\times 1{\cdot}0009$	$\times 1{\cdot}0009$
$\times 1{\cdot}00001$	$\times 1{\cdot}00001$
$\times 1{\cdot}000005$	$\times 1{\cdot}000005$
$\times 1{\cdot}0000002$	$\times 1{\cdot}0000002$
$\times 1{\cdot}00000001$	$\times 1{\cdot}00000001$
$\times 1{\cdot}000000007$	$\times 1{\cdot}000000007$
$\times 1{\cdot}0000000001$	$\times 1{\cdot}0000000001$
$\times 1{\cdot}00000000007$	$\times 1{\cdot}00000000007$
$\times 1{\cdot}000000000000$;	$\times 1{\cdot}000000000001$.*

From the form of these factors it appears that the table requisite for the application of the one-figure process must be one of $\log(1+{\cdot}1^m n)$, for values of n from 0 to 9, and values of m from 1 to 12. These logarithms admit of convenient arrangement in twelve columns, corresponding to the values of m, an additional column receiving the values of n, to serve as argument. By the aid of such a table, $\log\pi$, as arising from the two preceding operations, is formed as follows:—

301029995664		log 2	397940008672		colog 4
176091259056	5	Col. 1	79181246048	2	Col. 1
17033339299	4	„ 2	17033339299	4	„ 2
2597980720	6	„ 3	2597980720	6	„ 3
390689250	9	„ 4	390689250	9	„ 4
4342923	1	„ 5	4342923	1	„ 5
2171467	5	„ 6	2171467	5	„ 6
86859	2	„ 7	86859	2	„ 7
4343	1	„ 8	4343	1	„ 8
3040	7	„ 9	3040	7	„ 9
43	1	„ 10	43	1	„ 10
30	7	„ 11	30	7	„ 11
0	0	„ 12	0	1	„ 12
0·497149872694		$\log\pi$	0·497149872694		$\log\pi$

* It will be observed that the last eleven factors are the same in both sets. This arises from the *casual* circumstance that the figures of the products of the first two factors are the same in both: $2\times 1{\cdot}5=3$, and $\frac{1}{4}\times 1{\cdot}2={\cdot}3$.

Both results are true in the last figure.

I now propose the resolution of the same number, π, by the two-figure process.

The operation follows:—

```
                3 14 15 92 65 35 90 |÷3        31 41 59 26 53 59 0×6
               ---------------------|          -------------------
1·04           1·04 71 97 55 11 97  | 69      1·88)49 55 59 21 54 | 26
  62 4              62 4            |              37 6           |
                   -----            |             -----           |
                    9 57            |              11 95          |
  9 36              9 36            |              11 28          |
--------           -----            |             ------          |
1·04 71 76        )21 55 11 9       | ·20   1·88 48 88)67 59 21 5  | 35
        21         20 94 35 2       |                56 54 66 4   |
--------          ------------      |               -----------   |
1·04 71 97           )60 76 77      | 58              11 04 55 14 |
                      52 35 99      |                  9 42 44 40 |
                     ---------      |                 ----------- |
                       8 40 78      |       1·88 49 54 )1 62 10 74 | 86
                       8 37 76      |                   1 50 79 63 |
                      --------      |                  ----------  |
                         .3 02      | 02                  11 31 11 |
                          2 09      |                     11 30 97 |
                         -----      |                   ---------- |
                            93      | 89                        14 | 00 07
                            84      |                           13 |
                            --      |                           -- |
                             9      |                              |
```

Two resolutions are given, as before, using different preparing factors, and forming the divisors in the second resolution by subtraction. These operations do not require the same minuteness of description as the former. In consequence of the extension of the quotients to two figures, only three divisors have to be formed, and the operations then, as before, pass into contracted division. The two sets of factors are:—

$$\begin{array}{ll}
3\times 1{\cdot}04 & \text{and } \tfrac{1}{6}\times 1{\cdot}88 \\
\times 1{\cdot}00,69 & \times 1{\cdot}00,26 \\
\times 1{\cdot}00,00,20 & \times 1{\cdot}00,00,35 \\
\times 1{\cdot}00,00,00,58 & \times 1{\cdot}00,00,00,86 \\
\times 1{\cdot}00,00,00,00,02 & \times 1{\cdot}00,00,00,00,00,07. \\
\times 1{\cdot}00,00,00,00,00,89;
\end{array}$$

The orderly succession of the quotient figures from the decimal point is here seen to be the same as in the former case. And hence also, the table requisite for use in connexion with this process will be one of log $(1 + {\cdot}01^{m}n)$, for values of n from 00 to 99, and

values of m from 1 to 6. It will thus occupy six columns of 100 values each. The following shows the application of such a table to the above resolutions:—

47 71 21 25 47 20	log 3	22 18 48 74 96 16	colog 6
1 70 33 33 92 99	04 Col. 1	27 41 57 84 92 64	88 Col. 1
29 86 34 08 57	69 „ 2	11 27 70 02 77	26 „ 2
8 68 58 03	20 „ 3	15 20 00 41	35 „ 3
25 18 91	58 „ 4	37 34 93	86 „ 4
87	02 „ 5	00	00 „ 5
39	89 „ 6	3	07 „ 6
0·49 71 49 87 26 96		0·49 71 49 87 26 94	

Finally, I propose the resolution of π into factors by the three-figure process.

This is the process with which we were familiarised in the former paper. I resolve the given number in two ways, as follows:—

	3·141 592 653 590	$\div 2$	314 159 265 359 0	$\times 6$
1·570	$\cancel{1}\cdot\cancel{5}\cancel{7}\cancel{0}$ 796 326 795	507	1·884)955 592 154	507
785	785 0		942 0	
	11 326		13 592	
11	10 990		13 188	
1·570 79$\cancel{6}$	)336 795	214	1·884 95$\cancel{6}$)404 154	214
	314 159		376 991	
	22 636		27 163	
	15 708		18 850	
	6 928		8 313	
	6 283		7 540	
	645	410	773	410
	628		754	
	17		19	
	16		19	
	1		0	

Little needs now to be said in explanation. The number of factors in $1+N^1$ is reduced to four, and the number of divisors to two. As there remain, after the formation of n_2 ($=$·000,507), only three figures to be annexed to the remainder, it is unnecessary to make provision for an increase of more than the same number in

the figures of the divisor. The division then goes on to the close by periodical curtailment of the divisor.

Referring to the right-hand operation, the complete second divisor would be formed as follows:—

$$\begin{array}{r} 1{\cdot}884 \\ 942 \\ 13\ 188 \\ \hline 1{\cdot}884{,}955{,}188 \end{array} \qquad \text{or,} \qquad \begin{array}{r} 1{\cdot}884{,}955{,}592 \\ 404 \\ \hline 1{\cdot}884{,}955{,}188 \end{array}$$

Hence appears the reason of the direction in the former paper to increase the last figure of the contracted divisor by unit when the next figure of the complete divisor would be 5 or more.

The two sets of factors here are:—

$$\begin{array}{ll} 2\times 1{\cdot}570 & \text{and } \tfrac{1}{8}\times 1{\cdot}884 \\ \quad\times 1{\cdot}000{,}507 & \quad\times 1{\cdot}000{,}507 \\ \quad\times 1{\cdot}000{,}000{,}214 & \quad\times 1{\cdot}000{,}000{,}214 \\ \quad\times 1{\cdot}000{,}000{,}000{,}000{,}410; & \quad\times 1{\cdot}000{,}000{,}000{,}410.^{*} \end{array}$$

And the table here requisite is one that shall contain log $(1+{\cdot}001^{m}n)$ for values of n from 000 to 999, and for values of m from 1 to 4. Practically, it is unnecessary to tabulate the column corresponding to $m=4$, as the values belonging to it are those in the adjoining column with the last period cut off. The table here described is that which accompanied the former paper; and I apply it to the foregoing operations as follows:—

301 029 995 664	log 2
195 899 652 409	570 Col. 1
220 131 504	507 „ 2
92 939	214 „ 3
178	410 „ 4
0·497 149 872 694	

221 848 749 616	colog 6
275 080 898 457	884
220 131 504	507
92 939	214
178	410
0·497 149 872 694	

The reader is now in a position to judge of the comparative facility of the three modes of formation which have been exemplified, so far as the number of figures and of tabular entries requisite in each of them is concerned. Obviously the superiority in these respects rests with the last—the three-figure method. But it is in the resolving process that the superiority of this method chiefly resides, and it will soon be felt and admitted if the three modifications of the process are tried. The interruptions to the division for the formation of new divisors, when numerous, are attended by

* See Note, p. 127.

a degree of irksomeness which is not conducive to accuracy. The three-figure process, when applied to the formation of twelve-figure logarithms, has only one of these breaches of continuity, the formation of the divisor corresponding to which also requires hardly a thought; and when applied to the formation of nine-figure logarithms it has no breach at all (see *Ex.* 8, p. 81).

I am now to elucidate the converse operation—that of finding the number corresponding to a given logarithm. The process consists in decomposing the given logarithm by subtraction into a series of other logarithms contained in the tables, by which the factors of the required number become known. Multiplication of these factors then gives the required number. So far no demonstration is needed. It is necessary, however, to show that a single entry in the auxiliary table and in each column of the principal table will always suffice to exhaust the given logarithm; since on this circumstance depends the form of the *compounding* operation, as I call the multiplication of the factors.

The property of the tables just referred to belongs to them in virtue of the constitution of the several columns and their relation to each other. It belongs equally to all the tables, whether adapted to the one-figure, the two-figure, or the three-figure process; but I confine my illustrations to the last—the three-figure table—as the others are not before us.

For the present purpose we have to do with five columns, one —say the first—of the auxiliary table, and the four columns of the principal table. In the first of these columns the least term is log 2, and in each of the others I call that the least term which corresponds to 001. If for a moment we conceive log 10 (or 1) to be the least term in an imaginary column preceding the auxiliary column, then if to each of the five columns before us the least term in that immediately preceding were annexed, it would be found, that in each of the series so formed the difference between any two consecutive terms is less than the least term in that series. So that if from a logarithm intermediate in value to two consecutive terms in any of the series, the less of the two were subtracted, the remainder, being necessarily less than the difference between the two, would also be less than the least term in that series. Hence, generally, if with any given logarithm less than log 10, we enter the column which contains the greatest value not exceeding it, and subtract from it that value; if with the remainder we enter in like manner a subsequent column, subtracting as before, and so on, till we reach the last column, we should, if the loga-

rithm have not been exhausted before, succeed in exhausting it by an entry in that column, since it contains all values from the least in Column III. to 0. And thus, with in no case more than a single entry in each column, the given logarithm will be decomposed into a series of tabular logarithms; and the numbers corresponding to these being known, the factors of the required number are determined.

An entry in the auxiliary table is *necessary* only if the logarithm to be decomposed be not less than log 2; but we may, nevertheless, if we please, *always* commence with an entry in that table. For it will be found that, in the case of *any* given logarithm, the auxiliary table affords a choice of no fewer than five or six distinct logarithms, the subtraction of any one of which (indices being neglected), will leave a remainder less than log 2. The advantage of this is, that we have always the means of verifying our results by employing different methods of decomposition.

It is, as will be seen presently, the argument figures, corresponding to the logarithms into which any given logarithm is decomposed, that form the multipliers in the compounding process. Hence, the component logarithms being taken from successive columns (no two of them from the same column), it follows, from the relation of the values in the several columns to their corresponding arguments, that the multipliers descend in orderly succession from the decimal point: no two of them—of how many figures soever each may consist—can, so to speak, *overlap* each other. It is by the knowledge of this that we are guided in the arrangement of the compounding process.

I shall now give some examples of the formation of antilogarithms, illustrating, as before, the one, two, and three-figure processes.

Given $\log \pi = 0·497149872694$; required π.

497149872694	log 2	1·5	4
301029995664		60	
196119877030	5 Col. 1	1·560	6
176091259056		9360	
		1·569360	9
20028617974	4 " 2	14124240	
17033339299			
		1·5707724$\cancel{2}$40	1
		15707724	
2995278675	6 " 3		
2597980720		1·570788$\cancel{1}$31724	5
		7853941	
397297955	9 " 4	1·57079$\cancel{6}$985665	2

397297955	9	Col. 4
390689250		
6608705	1	„ 5
4342923		
2265782	5	„ 6
2171467		
94315	2	„ 7
86859		
7456	1	„ 8
4343		
3113	7	„ 9
3040		
73	1	„ 10
43		
30	7	„ 11
30		

1·57079~~5~~985665	2
314159	1
15708	7
10996	1
157	7
110	
1·570796326795	×2
3·141592653590	$=\pi$

The operation is as above, being the converse of one of those on page 125. The portion on the left shows the decomposition of the given logarithm into the following sum:—

$$\log 2+\log 1{\cdot}5+\log 1{\cdot}04+\log 1{\cdot}006+\log 1{\cdot}0009, \text{ and so on;}$$

and the portion on the right shows the compounding or multiplication of the corresponding factors. It is this latter portion alone that stands in need of explanatory remark.

The factors may, of course, be multiplied in any order, and we might have commenced with 2—the factor corresponding to the logarithm taken from the auxiliary table.* I prefer, generally, to leave this factor to the last, and to use up, first, those of the form $1+n$, and in the order in which they arise. Now, to multiply by a factor of this form, it suffices to add to the multiplicand its product by n; for $a(1+n)=a+an$. So, taking the factor 1·5 as the first multiplicand, we obtain the product of the first two factors by adding to this multiplicand its product by ·04, and thus form a second multiplicand. In like manner, using ·006 as a multiplier, we obtain the product of the first three factors, and so form a third multiplicand, and so on.

But it is convenient to disregard the ciphers which precede the

* See Example, p. 90.

significant figures in the several multipliers, as we can determine by other indications the position in the decimal scale which the several partial products ought to occupy. We know that the significant figures of the several multipliers descend regularly in the scale, and in the one-figure process the exact place of each corresponds with the number of the column placed against it. Thus, 5 is in the first decimal place, 4 in the second, and so on. Consequently, the last figure of the first multiplicand being in the first place, and the multiplier being in the second, the right-hand figure of their product will be in the third, since $1+2=3$. In like manner, the last figure of the second multiplicand being in the third place, and the corresponding multiplier being also in the third, the right-hand figure of their product will be in the sixth, since $3+3=6$. This principle will guide us as to the placing of the next partial product, the last figure of which will fall in the tenth place, since $6+4=10$. The following product, however, would extend beyond the twelfth place, which is the limit of our operation, and therefore we here apply the principle somewhat differently. The next multiplier, 1, is in the fifth place, and therefore, since $12-5=7$, it follows that the seventh decimal place of the multiplicand is that whose product will fall in the twelfth place. Contraction, therefore, commences at this point, and it will readily appear that each succeeding multiplicand will have one effective figure fewer than that which preceded it. After the formation of the sixth multiplicand, the operation passes into that of contracted multiplication, a figure being cut off from the multiplicand at each step.

Another and somewhat different method may be employed to ensure the proper placing of the products in the compounding operation. The first portion of the operation in question is repeated in the margin, to exemplify the process now referred to. The decimal places of the first multiplicand being made up to twelve by the annexation of ciphers, the orderly curtailment of it and the succeeding multiplicands in the manner shown, furnishes, it will readily be perceived, a very perspicuous guide for the properly placing of the several partial products. This method will probably be found superior to the other when paper ruled in squares is not used.

$1{\cdot}5000000000\cancel{0}0$	4
60000000000	
$1{\cdot}560000000\cancel{0}00$	6
9360000000	
$1{\cdot}56936000\cancel{0}000$	9
1412424000	
$1{\cdot}5707724\cancel{2}4000$	1
15707724	
$1{\cdot}570788\cancel{1}31724$	5
7853941	
$1{\cdot}57079\cancel{5}985665$	

I now propose the same example by the two-figure process:—

49 71 49 87 26 94	colog 6
22 18 48 74 96 16	
27 53 01 12 30 78	88 Col. 1
27 41 57 84 92 64	
11 43 27 38 14	26 „ 2
11 27 70 02 77	
15 57 35 37	35 „ 3
15 20 00 41	
37 34 96	86 „ 4
37 34 93	
3	07 „ 6
3	

1·88	26
37 6	
11 28	
1·88 48 88	35
56 54 66 4	
9 42 44 40	
1·88 49 5$\cancel{3}$ 97 10 80	86
1 50 79 63	
11 30 97	00 07
13	
1·88 49 55 59 21 53	÷6
3·14 15 92 65 35 88	=π

Above are the two portions of the operation, the second only of which, as in the former case, needs remark. The multipliers consist each of two figures, but the principle by which we were guided in placing the products in the previous example is equally applicable here. The last figure of the first multiplicand is in the second decimal place, and the first figure of the multiplier, 26, is in the third. Therefore, since $2+3=5$, the right-hand figure of the first product will fall in the fifth place, and that of the second product will fall in the sixth. In like manner, the right-hand figures of the next two products will fall in the eleventh and twefth places respectively ($6+5=11$, and $6+6=12$). The first figure of the third multiplier, 86, is in the seventh place; and hence, since $12-7=5$, the fifth decimal place of the multiplicand will be the last effective figure within our limit of twelve places. The process is now that of contracted multiplication, and the absence of a factor from Col. 5 will give no trouble.

Finally, I repeat the example by the three-figure process:—

497 149 872 694	log 2
301 029 995 664	
196 119 877 030	570 Col. 1
195 899 652 409	
220 224 621	507 „ 2
220 131 504	
93 117	214 „ 3
92 939	
178	410 „ 4
178	

1·570	507
785 0	
10 990	
1·570 79$\cancel{6}$ 990	214
314 159	
15 708	
6 283	410
628	
16	
1·570 796 326 794	×2
3·141 592 653 588	

The same principle is our guide in the placing of the products here. For the first product we have $3+4=7$, and for the second $3+6=9$, the latter falling *two* places in advance of the former, in consequence of the occurrence of a cipher. For the next we have $12-7=5$; that is, the fifth decimal place of the second multiplicand is the last effective figure, its product falling in the twelfth place. And here, also, the operation changes into that of contracted multiplication.

It is with a view to perspicuity that I have exemplified the three processes, as each of the last two follows from that which precedes it by a natural sequence. But it will be understood that it is the tables for the application of the three-figure process alone that accompany these papers. The superiority of the process last named over the others, in regard to the number of tabular entries and of figures employed, is equally apparent in the anti-logarithmic as in the logarithmic application. It will be still more fully appreciated after the trial of a few examples.

I must postpone till another opportunity a description of the methods employed for the construction and verification of the tables, and also an account of the origin of the processes that have been exemplified.

On Interpolation, Summation, and the Adjustment of Numerical Tables. (Part III.) By W. S. B. WOOLHOUSE, F.R.A.S., F.S.S., *Vice-President of the Institute of Actuaries, &c.*

[Read before the Institute, 27th February, 1865.]

THE ADJUSTMENT OF NUMERICAL TABLES.

WHEN a series of quantities which depend on a fixed law, whether known or implied, follow each other in a due order of succession, the general accuracy of their numerical values may be satisfactorily tested by observing the regularity of the progression of a suitable order of differences. If the tabular quantities are the results of calculation from a given formula, with equidistant arguments, by differencing them up to a certain order the existence of an isolated error, if one should exist, is thus prominently exhibited and therefore speedily detected. Moreover the precise locality of the error and the fact of its individuality being indicated by a characteristic interruption of the law of progression, the proper correction is ascertained by carefully noting the central position of the dis-

turbance, and revising the corresponding portion of the original computation. This work of revision may, however, be dispensed with, if necessary or desirable, as the magnitude as well as the locality of the indicated error can in general be readily deduced, with sufficient accuracy, directly from the differences, by the method of adjustment here given. The method, being so far independent of fundamental origin, is therefore equally applicable to the adjustment of a series of quantities derived from observation, or from statistical data; only in this latter case, as almost every quantity must be more or less affected by imperfections arising from defective information and other known or unknown incidental causes, the influence of which may sometimes be augmented by paucity of numbers, the progression of the differences is disturbed by a more complicated combination of the effects of an irregular series of errors, and it becomes more difficult to eliminate the value of each separate correction.

In constructing and adjusting numerical tables, more especially those drawn from statistics, the method commonly employed is simply that of taking as data only a few of the unadjusted tabular values, the same being distributed apart at certain assigned convenient intervals, and thence determining all the intermediate values by a suitable process of interpolation, so as to impart to the results a desirable graduated progression. Although expedient as regards facility of calculation, this practice is in other respects devoid of scientific principle. The meagre data upon which everything is made to depend may indeed consist of values amongst those most in error, and under any circumstances an undue and exclusive presumption of accuracy is tacitly given to a few values arbitrarily chosen to the exclusion of all the others; and by using these values without any preliminary adjustment, the entire table is necessarily and systematically vitiated by whatever imperfections they may happen to possess. The real problem to be solved is to determine a method of constructing a properly graduated table, which shall represent the whole of a given series of values, or a given mass of original facts, with a greater degree of accuracy than that of any other table of a similar kind. A thorough investigation of the subject, thus considered, and taken in all its bearings, may be said to open out an entirely new branch of science; and in the discussion given in the present paper it will, I think, be admitted that some progress has been made towards its mathematical development.

The uniform interval between the consecutive arguments of the

table being supposed to be sufficiently small, it will follow that if the exact functions were differenced, the differences would rapidly converge and exhibit no sign of discrepancy. But the given functions or quantities involve, in addition to the exact functions, their respective errors; and it is evident that the amounts of variation of the several differences would be produced by separately differencing the series of errors which, on account of their irregularity and consequent divergence of the differences, soon become prominently conspicuous.

ISOLATED ERRORS.

First consider the case of a single error. Let the value V_0 be in error so as to require a correction v, and let the values which immediately precede and follow it be supposed to be exact. Then the effects produced on the several orders of differences by the correction v will be as follows:—

Correction.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
0		--		--		--		$+v$
	--		--		--		$+v$	
0		--		--		$+v$		$-8v$
	--		--		$+v$		$-7v$	
0		--		$+v$		$-6v$		$+28v$
	--		$+v$		$-5v$		$+21v$	
0		$+v$		$-4v$		$+15v$		$-56v$
	$+v$		$-3v$		$+10v$		$-35v$	
...v		$-2v$		$+6v$		$-20v$		$+70v$
	$-v$		$+3v$		$-10v$		$+35v$	
0		$+v$		$-4v$		$+15v$		$-56v$
	--		$-v$		$+5v$		$-21v$	
0		--		$+v$		$-6v$		$+28v$
	--		--		$-v$		$+7v$	
0		--		--		$+v$		$-8v$
	--		--		--		$-v$	
0		--		--		--		$+v$

These are simply deduced by successive differencing in an expanded form, the absence of a symbol and the insertion of hyphens being always accounted as zero; and it will be observed that the numerical coefficients which appertain to each order n are those of the expansion of the binomial $(1-1)^n$.

Retaining the notation before employed ("Interpolation," vol. xi., page 64), the corresponding portion of the differences of the given series of quantities will stand thus:—

Given Values.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
V_{-4}								h_{-4}
							g_{-4}	
V_{-3}						f_{-3}		h_{-3}
					e_{-3}		g_{-3}	
V_{-2}				d_{-2}		f_{-2}		h_{-2}
			c_{-2}		e_{-2}		g_{-2}	
V_{-1}		b_{-1}		d_{-1}		f_{-1}		h_{-1}
	a_{-1}		c_{-1}		e_{-1}		g_{-1}	
...V_0		b_0		d_0		f_0		h_0
	a_{+1}		c_{+1}		e_{+1}		g_{+1}	
V_{+1}		b_{+1}		d_{+1}		f_{+1}		h_{+1}
			c_{+2}		e_{+2}		g_{+2}	
V_{+2}				d_{+2}		f_{+2}		h_{+2}
					e_{+3}		g_{+3}	
V_{+3}						f_{+3}		h_{+3}
							g_{+4}	
V_{+4}								h_{+4}

Now the quantity V_0, when corrected, will be V_0+v; and the adjustments produced by v on these differences will in like manner be obtained by incorporating the respective values contained in the former table.

To detect the existence, as well as to determine the value of the correction v, it will be most convenient to have recourse to an even order of differences, as the greatest disturbance is then exhibited in the central difference, or that which stands opposite to V_0, the quantity in error. The practical methods of calculation which follow will now be obvious.

Determination of an Error from Second Differences.

An error in the value of a quantity V_0 will be indicated by a marked disturbance of three second differences, b_{-1}, b_0, b_{+1}, and the required correction is to be inferred so as to reduce them to an arithmetical progression. Thus, when the corrected second differences $b_{-1}+v$, b_0-2v, $b_{+1}+v$, are in arithmetical progression, the two third differences, $c_{-1}-3v$, $c_{+1}+3v$, derived from them, will be equal, and consequently the fourth difference, d_0+6v, will vanish or become neutralized;

$$\therefore \text{ the correction } (v) = -\frac{d_0}{6} = \frac{(b_0-b_{-1})+(b_0-b_{+1})}{6}.$$

Hence the following simple rule, which may in general be applied mentally or by mere inspection, it being, of course, always understood that proper attention is given to the algebraic signs + and −.

RULE.—*From the middle second difference subtract separately the preceding and following differences; add together the two remainders, and divide by 6. The result, with its proper algebraic sign, will be the correction to be made in the value which stands opposite to the middle difference.*

Example.—To test the accuracy of a consecutive series of tabular numbers, they are differenced as follows:—

Tabular Numbers.	First Differences.	Second Differences.	Corrections of Second Differences.	Adjusted Second Differences.
1728				
	+469			
2197		+ 78		
	547			
2744		84		
	631			
3375		103*	−13 . . . v	+90
	734			
*4109		70*	+26 . . . $-2v$	96
	804			
4913		115*	−13 . . . v	+102
	919			
5832		108		
	1027			
6859		+114		
	+1141			
8000		&c.		
	&c.			
&c.				

Here the three second differences distinguished by an asterisk are decidedly irregular as regards progression, and show that the tabular number opposite to the central difference (+70) must be in error. And for the adjustment of this error, according to the rule, we have

Irregular	+103	−33	middle—preceding diff.
Second	70	−45	middle—following diff.
Differences	+115		
		6)−78	
Correction v		−13	
V		4109	
Adjusted value		4096	

By replacing 4109 by this adjusted value, and revising the differences, they will now observe a regular progression, showing that no other error exists amongst the numbers put down. The second differences may, however, be otherwise directly and independently adjusted by applying to their original values the respective corrections $+v$, $-2v$, $+v$, which is already done at the right hand of the above table.

The tabular numbers quoted in this example are the cubes of the numbers 12, 13, 14, ... 20. By differencing them out to fourth differences, even a single unit in error is at once disclosed; thus:—

True Values.				
Numbers.	Differences.			
1728	+ 469			
2197	547	+ 78		
2744	631	84	+6	0
3375	721	90	6	0
4096	817	96	6	0
4913	919	102	6	0
5832	1027	108	6	0
6859	+1141	+114	+6	
8000				

Primary Error of an Unit.				
Numbers.	Differences.			
1728	+ 469			
2197	547	+ 78		
2744	631	84	+6	+1
3375	722	91	7	−4
*4097	816	94	3	+6
4913	919	103	9	−4
5832	1027	108	5	+1
6859	+1141	+114	+6	
8000				

The differences $(+1, -4, +6, -4, +1)$ here agree in character with the fourth difference column of the first table, and thereby show that the indicated error is the only one that exists.

Determination of an Error from Fourth Differences.

The value of the correction of a quantity may in like manner be deduced from three disturbed fourth differences, so as to fulfil the condition that the three differences when adjusted shall form an arithmetical progression. The corrected fourth differences are here $d_{-1}-4v$, d_0+6v, $d_{+1}-4v$; and when these are in arithmetical progression, the two fifth differences, $e_{-1}+10v$, $e_{+1}-10v$, derived from them, will be equal, and therefore the sixth difference, f_0-20v, will vanish or become neutralized;

$$\therefore \text{ the correction } (v)=\frac{f_0}{20}=\frac{(d_{-1}-d_0)+(d_{+1}-d_0)}{20},$$

which suggests the following rule:—

Rule.—*From the preceding and following fourth differences separately subtract the middle difference; add together the two remainders, and divide by 20. The quotient, with its proper algebraic sign, will be the required correction, and will be applicable to the quantity which stands opposite to the middle difference.*

By the foregoing table, showing the effects produced by v on the differences, it will appear that five fourth differences will be subjected to alteration, and that they may be adjusted, with respect to v, without redifferencing, by directly applying to them the several corrections $+v$, $-4v$, $+6v$, $-4v$, $+v$. If the differences so corrected, when taken in connexion with those that immediately precede and follow them, should then be found to observe a satisfactory progression, the fact will show that the primitive values immediately adjoining V_0 are free from material error. But if any marked irregularities should yet remain, other corrections will

require to be made in a similar manner; and by thus continuing to repeat the process at points of greatest derangement, all the more prominent disturbing inequalities of the given series of numbers may be ultimately regulated.

Whether second or fourth differences should be used in these calculations for the adjustment of any proposed set of numbers, must, to a certain extent, be left to the practical judgment of the computer. Perhaps fourth differences will be found to be the more generally applicable. As a general guide, however, it may be advisable first to consider what even order of differences of exact functions ought to be small enough to be practically disregarded, and then to employ differences of the next less even order. And here it will not pass unnoticed that the foregoing simple rules of calculation may, if required, be readily adapted to higher orders of differences.

The rules, as they are here given, were, many years ago, uniformly and successfully employed by myself in officially examining and testing the accuracy of the results of the various and extensive computations made in the Nautical Almanac Office; and I also availed myself of their application in the adjustment of both the "Indian Military" and the "Experience" tables of mortality.* In the construction of all kinds of tables these rules will in general be found to be very effective in the adjustment of isolated or prominent irregularities, as they proceed upon the sound principle of applying each successive correction at the very point where the probable or certain existence of an error is most strongly indicated. If, for example, a number of errors, each of which may be of any magnitude, were promiscuously introduced into a page of a table of logarithms, or in any table computed mathematically according to a given formula or generic law which is still inherent to the general mass of values, the preceding rules, when applied to the differences, would necessarily have the effect of practically eliminating the several errors and restoring the table to its original state of accuracy. In all ordinary calculations involving the construction and adjustment of tables what has already been laid down may therefore be considered to meet every necessary requirement.

* "*Investigation of Mortality in the Indian Army.* By W. S. B. Woolhouse, F.R.A.S., &c.," and "*Tables Exhibiting the Law of Mortality deduced from the Combined Experience of Seventeen Life Assurance Offices.*" It is presumed that no existing graduated table of mortality represents so closely the general body of facts on which it is founded as the tables here referred to. Mr. Jellicoe, the President of the Institute, at the end of a valuable paper inserted in vol. i., page 166, has given a D and N table, together with the values of an annuity and annual premium, calculated from the adjusted Bengal Military Mortality, at 4 per cent.

Multiform Errors.

We have yet, however, to enter upon a discussion of another part of the subject, which, if it be less essential in ordinary practice, nevertheless admits of most important applications, and also possesses considerable interest in a mathematical point of view. If the generic law be unknown, and the table—as in the case of a table of mortality—be drawn from observation, then no part of it can be predicated as really or even probably accurate, and therefore, after the more prominent displacements and straggling errors have been adjusted as before described, it may afterwards be requisite, if a continuous and accurately-finished table be desired, to have recourse to some other more recondite process in order to finally eradicate a consecutive series of lesser imperfections, the disturbing effects of which on the differences are of course blended together without exhibiting any individual traces.

To proceed to an investigation of the subject in this more complicated form, we have now to discuss the question of the complete adjustment of a combination of multiform small errors, or the neutralization of the assigned discrepancies of an irregular series of differences of a given order.

In the first place we shall consider the case in relation to fourth differences. Let the given original quantities be denoted by $V_0, V_1, V_2, \ldots. V_n$, and the sought corrections, with their proper algebraic signs, respectively by $v_0, v_1, v_2, \ldots. v_n$; and suppose them to be separately differenced as follows:—

$$
\begin{array}{l|l|l|l|l}
V_0 & & & & \\
 & a_0 & & & \\
V_1 & & b_0 & & \\
 & a_1 & & c_0 & \\
V_2 & & b_1 & & d_0 \\
 & a_2 & & c_1 & \\
V_3 & & b_2 & & d_1 \\
 & a_3 & & c_2 & \vdots \\
V_4 & & b_3 & \vdots & \vdots \\
 & a_4 & \vdots & \vdots & d_{n-4} \\
V_5 & \vdots & \vdots & c_{n-3} & \\
\vdots & \vdots & b_{n-2} & & \\
 & a_{n-1} & & & \\
V_n & & & &
\end{array}
\qquad\qquad
\begin{array}{l|l|l|l|l}
v_0 & & & & \\
 & a_0 & & & \\
v_1 & & \beta_0 & & \\
 & a_1 & & \gamma_0 & \\
v_2 & & \beta_1 & & \delta d_0 \\
 & a_2 & & \gamma_1 & \\
v_3 & & \beta_2 & & \delta d_1 \\
 & a_3 & & \gamma_2 & \vdots \\
v_4 & & \beta_3 & \vdots & \vdots \\
 & a_4 & \vdots & \vdots & \delta d_{n-4} \\
v_5 & \vdots & \vdots & \gamma_{n-3} & \\
\vdots & \vdots & \beta_{n-2} & & \\
 & a_{n-1} & & & \\
v_n & & & &
\end{array}
$$

Then $V_0+v_0, V_1+v_1, \ldots. V_n+v_n$ are the corrected functions, and their corresponding fourth differences will evidently be $d_0+\delta d_0, d_1+\delta d_1, \ldots. d_{m-4}+\delta d_{m-4}$, where δ is employed as a characteristic of variation, and not as a quantity. In most cases it may be practicable to assign such small values to $\delta d_0, \delta d_1, \ldots. \delta d_{n-4}$ as shall impart a tolerable progression to these cor-

rected differences, and thus neutralize the principal irregularities of the given differences $d_0, d_1, \ldots . d_{n-4}$. The values $\delta d_0, \delta d_1$, &c., thus found, will be the only data for the determination of $v_0, v_1, \ldots . v_n$; and as the number of these data, or mathematical conditions, is here less by four than the number of unknown values, the problem remains so far essentially indeterminate. This is indeed evident from the consideration that by arbitrarily assuming any four numbers for the unknown leading differences v_0, a_0, β_0, γ_0, and using these numbers in combination with the assigned fourth differences $\delta d_0, \delta d_1, \ldots . \delta d_{n-4}$, the several values of v, in common with the whole scheme of differences, become directly deducible by summation. The gist of the present inquiry is to assign such values to v_0, a_0, β_0, γ_0, that the resulting displacements, $v_0, v_1, \ldots v_n$, of the original functions, shall be as small as possible. To effect this object the general amount of disturbance is most satisfactorily estimated by adding together the squares of the several corrections, or errors, for by that means positive and negative disturbances, which are alike objectionable, become all of them positive, and therefore accumulative. Hence the problem, mathematically enunciated, resolves itself into the following :—

PROBLEM.—*To find a series of small quantities, $v_0, v_1, \ldots . v_n$, which shall have a given set of fourth differences, and such that the sum of their squares shall be a minimum.*

Before proceeding with the investigation of this problem, it may not be out of place here to premise a brief outline of the practical resolution of equations of condition by Gauss's "method of least squares," as the same may not be generally known or understood. At least, I believe it has not before been explained on elementary algebraical principles.

Let a series of errors be expressed by the linear equations

$$\begin{aligned} e_0 &= a_0x+b_0y+c_0z \ \text{ \&c.} \\ e_1 &= a_1x+b_1y+c_1z \ \text{ \&c.} \\ e_2 &= a_2x+b_2y+c_2z \ \text{ \&c.} \\ &\text{\&c.} \qquad\qquad \text{\&c.} \end{aligned}$$

where a_0, b_0, c_0, &c. are given coefficients; let the number of equations exceed the number of variables x, y, z, &c.; and let it be required to find values of these variables, such that the sum of the squares of the errors, or

$$S=e_0^2+e_1^2+e_2^2+\text{\&c.}$$

shall be a minimum.

When this last condition subsists, any small alteration, whether

positive or negative, in the values of one or more of the variables, must increase the value of S. Suppose x to become $x+\varepsilon$; then the series of errors will evidently become $e_0+a_0\varepsilon$, $e_1+a_1\varepsilon$, $e_2+a_2\varepsilon$, &c., and therefore the sum of their squares will be

$$S+2(a_0e_0+a_1e_1+a_2e_2\ldots\ldots)\varepsilon+(a_0^2+a_1^2+a_2^2\ldots\ldots)\varepsilon^2.$$

Here the third term, involving ε^2, being essentially positive, it is evident that the value of the expression will always exceed S, whether ε be positive or negative, provided that

$$a_0e_0+a_1e_1+a_2e_2 \text{ \&c.}=0.$$

And in a similar manner, by separately varying y, z, &c., it may be inferred that the same condition will be further maintained when

$$\begin{array}{l} b_0e_0+b_1e_1+b_2e_2 \text{ \&c.}=0 \\ c_0e_0+c_1e_1+c_2e_2 \text{ \&c.}=0 \\ \quad\text{\&c.} \qquad\quad \text{\&c.} \end{array}$$

If in each of these equations the errors e_0, e_1, e_2, &c., be replaced respectively by the preceding expressions, they will serve for the determination of the variables. Hence the following rule:—

Rule.—*Multiply the equations severally by the respective coefficients of one particular variable, and equate the sum of the products with zero. By proceeding thus with respect to each separate variable, as many equations as variables are obtained; and by solving these equations, of the first degree, the values of the several variables are ultimately determined.*

To apply this method of least squares to the present inquiry it will be requisite to express the general value (v_m) in terms of v_0, α_0, β_0, γ_0, and the given fourth differences $\delta d_0, \delta d_1, \ldots\ldots \delta d_{n-4}$. This will be readily effected by summation. Thus, we have evidently

$$\begin{aligned}
\gamma_0 &= \gamma_0 \\
\gamma_1 &= \gamma_0+\delta d_0 \\
\gamma_2 &= \gamma_0+\delta d_0+\delta d_1 \\
\gamma_3 &= \gamma_0+\delta d_0+\delta d_1+\delta d_2 \\
&\vdots \\
\gamma_{m-3} &= \gamma_0+\delta d_0+\delta d_1+\delta d_2\ldots\ldots+\delta d_{m-4} \\
\therefore\ \beta_{m-2} &= \beta_0+\Sigma\gamma \\
&= \beta_0+(m-2)\gamma_0+(m-3)\delta d_0+(m-4)\delta d_1 \text{ \&c.} \\
&= \beta_0+(m-2)\gamma_0+\Sigma_x\{(m-\overline{x+3})\delta d_x\}.
\end{aligned}$$

And, bearing in mind, the usual formula for the summation of a common factorial, viz.,

$$\Sigma_{1\ldots n} p(p+1)\ldots(p+n) = \frac{p(p+1)\ldots(p+n+1)}{n+2} \quad \ldots (f)$$

we similarly find

$$\begin{aligned} a_{m-1} &= a_0 + \Sigma\beta \\ &= a_0 + (m-1)\beta_0 + \frac{(m-2)(m-1)}{2}\gamma_0 \\ &\qquad + \Sigma\frac{(m-\overline{x+3})(m-\overline{x+2})}{2}\delta d_x; \end{aligned}$$

and $\therefore$

$$\begin{aligned} v_m &= v_0 + \Sigma a \\ &= v_0 + ma_0 + \frac{(m-1)m}{2}\beta_0 + \frac{(m-2)(m-1)m}{2.3}\gamma_0 \\ &\qquad + \Sigma\frac{(m-\overline{x+3})(m-\overline{x+2})(m-\overline{x+1})}{2.3}\delta d_x \quad \ldots (\delta) \end{aligned}$$

where Σ indicates a summation with respect to x from 0 to $m-4$.

When m takes the consecutive values 0, 1, 2 ... n, the formula (δ) will express with mathematical accuracy the several values of the series of corrections, and as the coefficients of the four variables v_0, a_0, β_0, γ_0, are respectively 1, m, $\frac{m(m-1)}{2}$, $\frac{m(m-1)(m-2)}{2.3}$, the resolving equations, by the method of least squares, so as to make Σv^2 a minimum, will be, according to the foregoing rule,

$$\Sigma v_m = 0,\ \ \Sigma m v_m = 0,\ \ \Sigma\frac{m(m-1)}{2}v_m = 0,\ \ \Sigma\frac{m(m-1)(m-2)}{2.3}v_m = 0;$$

and these are evidently equivalent to

$$\Sigma v_m = 0,\ \ \Sigma m v_m = 0,\ \ \Sigma m^2 v_m = 0,\ \ \Sigma m^3 v_m = 0 \quad \ldots . (e).$$

In effecting these summations with respect to m, from 0 to n, after substituting the formula (δ), the term involving δd_x will, in each case, consist of factorials, by further substituting the algebraical equivalents,

$$\begin{aligned} m &= (m-x) + x \\ m^2 &= (m-x)(m-\overline{x-1}) + (2x-1).(m-x) + x^2 \\ m^3 &= (m-x)(m-\overline{x-1})(m-\overline{x-2}) + 3(x-1).(m-x)(m-\overline{x-1}) \\ &\qquad + (3x^2-3x+1).(m-x) + x^3. \end{aligned}$$

To abbreviate the process, let

$$\mu_x = \frac{(n-\overline{x+3})(n-\overline{x+2})(n-\overline{x+1})(n-x)}{2.3}\delta d_x \quad \ldots (g);$$

also,

$$\begin{aligned}
M &= \Sigma\mu_x \\
P &= \Sigma\mu_x\{4n+(x+4)\} \\
Q &= \Sigma\mu_x\{10n^2+(4x+18)n+(x^2+6x+8)\} \\
R &= \Sigma\mu_x\{20n^3+(10x+50)n^2+(4x^2+26x+38)n \\
&\qquad +(x^3+8x^2+18x+8)\}.
\end{aligned}$$

Then, for all consecutive values of m from 0 to n, the results of the respective summations contained in (e) will be

$$(n+1)v_0+\frac{(n+1)n}{2}\alpha_0+\frac{(n+1)n(n-1)}{2.3}\beta_0 + \frac{(n+1)n(n-1)(n-2)}{2.3.4}\gamma_0+\frac{M}{4}=0,$$

$$\frac{(n+1)n}{2}v_0+\frac{(n+1)n(2n+1)}{2.3}\alpha_0+\frac{(n+1)n(n-1)(3n+2)}{2.3.4}\beta_0 + \frac{(n+1)n(n-1)(n-2)(4n+3)}{2.3.4.5}\gamma_0+\frac{P}{4.5}=0,$$

$$\frac{(n+1)n(2n+1)}{2.3}v_0+\frac{(n+1)^2n^2}{4}\alpha_0+\frac{(n+1)n(n-1)(12n^2+15n+2)}{2.3.4.5}\beta_0 + \frac{(n+1)n(n-1)(n-2)(10n^2+14n+3)}{3.4.5.6}\gamma_0+\frac{Q}{3.4.5}=0,$$

$$\frac{(n+1)^2n^2}{4}v_0+\frac{(n+1)n(2n+1)(3n^2+3n-1)}{2.3.5}\alpha_0 + \frac{(n+1)n(n-1)(10n^3+18n^2+5n-2)}{2.3.4.5}\beta_0 + \frac{(n+1)n(n-1)(n-2)(20n^3+40n^2+16n-3)}{4.5.6.7}\gamma_0+\frac{R}{4.5.7}=0.$$

Dividing these equations by the respective coefficients of v_0, they become

$$v_0+\frac{n}{2}\alpha_0+\frac{n(n-1)}{2.3}\beta_0+\frac{n(n-1)(n-2)}{2.3.4}\gamma_0+\frac{M}{4(n+1)}=0,$$

$$v_0+\frac{2n+1}{3}\alpha_0+\frac{(n-1)(3n+2)}{3.4}\beta_0 + \frac{(n-1)(n-2)(4n+3)}{3.4.5}\gamma_0+\frac{P}{10n(n+1)}=0,$$

$$v_0+\frac{3n(n+1)}{2(2n+1)}\alpha_0+\frac{(n-1)(12n^2+15n+2)}{4.5(2n+1)}\beta_0 + \frac{(n-1)(n-2)(10n^2+14n+3)}{3.4.5(2n+1)}\gamma_0+\frac{Q}{10n(n+1)(2n+1)}=0,$$

$$v_0+\frac{2(2n+1)(3n^2+3n-1)}{15n(n+1)}a_0+\frac{(n-1)(10n^3+18n^2+5n-2)}{30n(n+1)}\beta_0$$
$$+\frac{(n-1)(n-2)(20n^3+40n^2+16n-3)}{210n(n+1)}\gamma_0+\frac{\mathrm{R}}{35n^2(n+1)^2}=0.$$

Differencing these equations, and then dividing by the respective coefficients of a_0, we get

$$a_0+\frac{n-1}{2}\beta_0+\frac{3(n-1)(n-2)}{20}\gamma_0=\frac{15n\mathrm{M}-6\mathrm{P}}{10n(n+1)(n+2)}$$
$$a_0+\frac{3n-1}{5}\beta_0+\frac{n(n-2)}{5}\gamma_0=\frac{6(2n+1)\mathrm{P}-6\mathrm{Q}}{10(n-1)\ldots\ldots(n+2)}$$
$$a_0+\frac{4n^3+3n^2-n-2}{2(3n^2+3n+2)}\beta_0+\frac{(n-2)(10n^3+12n^2+n-3)}{14(3n^2+3n+2)}\gamma_0$$
$$=\frac{21n(n+1)\mathrm{Q}-6(2n+1)\mathrm{R}}{7(n-1)\ldots\ldots(n+2)(3n^2+3n+2)}.$$

Again differencing, and dividing by the coefficients of β_0,

$$\beta_0+\frac{n-2}{2}\gamma_0=-\frac{15\mathrm{M}}{(n+1)(n+2)(n+3)}+\frac{18n\mathrm{P}-6\mathrm{Q}}{(n-1)\ldots\ldots(n+3)}$$
$$\beta_0+\frac{4n-5}{7}\gamma_0=-\frac{6(3n^2+3n+2)\mathrm{P}-12(2n+1)\mathrm{Q}}{(n-2)\ldots\ldots(n+3)}-\frac{60\mathrm{R}}{7(n-2)\ldots(n+3)}$$
$$\therefore\ \gamma_0=\frac{210\mathrm{M}}{(n+1)\ldots\ldots(n+4)}-\frac{84(6n^2-3n+2)\mathrm{P}-420n\mathrm{Q}+120\mathrm{R}}{(n-2)\ldots\ldots(n+4)}$$
$$\beta_0=-\frac{30(4n-5)\mathrm{M}}{(n+1)\ldots\ldots(n+4)}+\frac{6(45n^2-9n+14)\mathrm{P}-24(9n+1)\mathrm{Q}+60\mathrm{R}}{(n-1)\ldots\ldots(n+4)}$$
$$a_0=\frac{30(n^2-n+1)\mathrm{M}}{(n+1)\ldots\ldots(n+4)}-\frac{6(20n^2+5n+8)\mathrm{P}-6(15n+4)\mathrm{Q}+24\mathrm{R}}{2.n\ldots\ldots(n+4)}$$
$$v_0=-\frac{2(2n+1)(n^2+n+3)\mathrm{M}-(6n^2+6n+5)\mathrm{P}+2(2n+1)\mathrm{Q}-\mathrm{R}}{(n+1)\ldots\ldots(n+4)}.$$

Hence, by restoring the foregoing values of M, P, Q, R, we find

$$v_0=\ \Sigma\frac{(x+1)(x+2)(x+3)}{(n+1)(n+2)(n+3)(n+4)}\mu_x$$
$$a_0=-3\Sigma\frac{(x+1)(x+2)\{n+4(x+4)\}}{n(n+1)(n+2)(n+3)(n+4)}\mu_x$$
$$\beta_0=\ \ 6\Sigma\frac{(x+1)\{n^2+(4x+15)n+(10x^2+66x+104)\}}{(n-1)n\ldots\ldots(n+4)}\mu_x$$
$$\gamma_0=-6\Sigma\frac{n^3+(4x+13)n^2+(10x^2+58x+74)n+(20x^3+160x^2+388x+272)}{(n-2)(n-1)n\ldots\ldots(n+4)}\mu_x.$$

These results are perhaps more systematic when put in the following form :—

$$v_0 = \quad \Sigma\frac{\mu(x+1)(x+2)(x+3)}{(n+1)\ldots\ldots(n+4)}$$

$$a_0 = -3\Sigma\frac{\mu(x+1)(x+2)}{(n+1)\ldots\ldots(n+4)}\left(1+4\frac{x+4}{n}\right)$$

$$\beta_0 = \quad 6\Sigma\frac{\mu(x+1)}{(n+1)\ldots\ldots(n+4)}\left(1+4\frac{x+4}{n}+10\frac{(x+4)(x+3)}{n(n-1)}\right)$$

$$\gamma_0 = -6\Sigma\frac{\mu}{(n+1)\ldots\ldots(n+4)}\left(1+4\frac{x+4}{n}+10\frac{(x+4)(x+3)}{n(n-1)}\right.$$
$$\left.+20\frac{(x+4)(x+3)(x+2)}{n(n-1)(n-2)}\right)\ldots\ldots(h).$$

From the values of v_0, a_0, β_0, γ_0, thus determined, the several values of v can be successively deduced by summation according to the scheme of differences, or they may be calculated separately and independently by substitution in the formula (δ).

To thus extend the investigation to the adjustment of higher orders of differences would involve the discussion of equations which increase both in number and complexity, and would evidently be a work of considerable labour. I have, however, by a somewhat novel artifice, succeeded in reducing the fundamental expressions to a form so remarkably unique and comprehensive, that the general solution of the problem, for any order of differences, becomes not merely simplified but divested of all complication whatever.

The primary basis of this improvement consists in prefixing to the scheme a certain set of supplementary differences obtained by a receding process in which the prefixed values of the original function are severally assumed to be zero. These supplementary differences will be placed in the form of a wedge, and will cause the differences of each order to be equal in number to that of the original functions, and impart to them the peculiar property of making the values of the several functions depend exclusively on any stated order of differences according to a formula in which the coefficients are all included in one common factorial expression.

Thus, resuming the case of fourth differences, and retaining an uniformity of notation, the extended scheme will become as follows, where the upper ridge of values v_0, a_{-1}, β_{-2}, γ_{-3}, δd_{-4}, are necessarily equal, though designated by different symbols:—

$$\begin{array}{c|c|c|c|c}
 & & & & \delta d_{-4} \\
 & & & \gamma_{-3} & \\
 & & \beta_{-2} & & \delta d_{-3} \\
 & \alpha_{-1} & & \gamma_{-2} & \\
v_0 & & \beta_{-1} & & \delta d_{-2} \\
 & \alpha_0 & & \gamma_{-1} & \\
v_1 & & \beta_0 & & \delta d_{-1} \\
 & \alpha_1 & & \gamma_0 & \\
v_2 & & \beta_1 & & \delta d_0 \\
 & \alpha_2 & & \gamma_1 & \\
v_3 & & \beta_2 & & \delta d_1 \\
 & \alpha_3 & & \gamma_2 & \\
v_4 & & \beta_3 & & \vdots \\
 & \alpha_4 & & \vdots & \\
v_5 & \vdots & \vdots & & \delta d_{n-4} \\
\vdots & \vdots & & \gamma_{n-3} & \\
\vdots & & \beta_{n-2} & & \\
 & \alpha_{n-1} & & & \\
v_n & & & &
\end{array}$$

Also the formula (δ), when made to commence with δd_{-4}, so as to include the four supplemental differences, will become simply

$$v_m = \Sigma' \frac{(m-\overline{x+3})(m-\overline{x+2})(m-\overline{x+1})}{2.3} \delta d_x \quad . \; . \; . \; . \quad (\epsilon).$$

For the coefficient of δd_x being a function of $m-x$, is independent of the epoch; and when the epoch is transferred to v_{-4}, the values of v_0, a_0, β_0, γ_0, are $v_{-4}=0$, $a_{-4}=0$, $\beta_{-4}=0$, $\gamma_{-4}=0$. The terms comprehended under Σ' (accentuated) will now begin with $x=-4$ and end with $x=m-4$.

When Σv_m^2 is a minimum, it has already been shown, equations (e), that

$$\Sigma v_m=0, \;\; \Sigma m v_m=0, \;\; \Sigma m^2 v_m=0, \;\; \Sigma m^3 v_m=0.$$

And as these, when compounded with any finite functions of x, must still vanish, they are evidently equivalent to the following:—

$$\begin{aligned}
&\Sigma v_m=0 \\
&\Sigma v_m(m-x)=0 \\
&\Sigma v_m(m-x)(m-\overline{x-1})=0 \\
&\Sigma v_m(m-x)(m-\overline{x-1})(m-\overline{x-2})=0.
\end{aligned}$$

When the value of v_m, by (ϵ), is substituted, the several coefficients of δd_x are now of the common factorial form (f); and if, as before, we abbreviate by putting

$$\mu_x = \frac{(n-\overline{x+3})(n-\overline{x+2})(n-\overline{x+1})(n-x)}{2.3} \delta d_x \quad . \; . \; . \quad (g)$$

the summations, after being taken with respect to all consecutive values of m from 0 to n, will give

$$\begin{aligned}
&\Sigma' \mu_x=0 \\
&\Sigma' \mu_x(n-\overline{x-1})=0 \\
&\Sigma' \mu_x(n-\overline{x-1})(n-\overline{x-2})=0 \\
&\Sigma' \mu_x(n-\overline{x-1})(n-\overline{x-2})(n-\overline{x-3})=0.
\end{aligned}$$

And these are again equivalent to

$$\left.\begin{array}{l}\Sigma'\mu_x\ \ =0\\ \Sigma'x\mu_x=0\\ \Sigma'x^2\mu_x=0\\ \Sigma'x^3\mu_x=0\end{array}\right\}\ .\ .\ .\ .\ .\ .\ .\ (\theta).$$

Hence, when Σ (unaccentuated) does not include values which depend on the four supplementary differences, these last equations will give

$$\begin{aligned}\Sigma\mu_x\ &=-\ \mu_{-4}-\ \mu_{-3}-\ \mu_{-2}-\mu_{-1}\\ \Sigma x\mu_x &=\ \ 4\mu_{-4}+\ 3\mu_{-3}+2\mu_{-2}+\mu_{-1}\\ \Sigma x^2\mu_x &=-16\mu_{-4}-\ 9\mu_{-3}-4\mu_{-2}-\mu_{-1}\\ \Sigma x^3\mu_x &=\ \ 64\mu_{-4}+27\mu_{-3}+8\mu_{-2}+\mu_{-1}\end{aligned}$$

for the determination of the four unknown values μ_{-4}, μ_{-3}, μ_{-2}, μ_{-1}, and thence δd_{-4}, δd_{-3}, δd_{-2}, δd_{-1}.

Adding each respective equation to the next following one, we get

$$\begin{aligned}\Sigma(x+1)\mu_x\ &=\ \ 3\mu_{-4}+\ 2\mu_{-3}+\ \mu_{-2}\\ \Sigma(x+1)x\mu_x &=-12\mu_{-4}-\ 6\mu_{-3}-2\mu_{-2}\\ \Sigma(x+1)x^2\mu_x &=\ \ 48\mu_{-4}+18\mu_{-3}+4\mu_{-2};\end{aligned}$$

and, adding twice each of these to the next following,

$$\begin{aligned}\Sigma(x+1)(x+2)\mu_x\ &=-6\mu_{-4}-2\mu_{-3}\\ \Sigma(x+1)(x+2)x\mu_x &=\ 24\mu_{-4}+6\mu_{-3};\end{aligned}$$

$$\therefore\ \mu_{-4}=\Sigma\frac{(x+1)(x+2)(x+3)}{2.3}\mu_x$$

$$\mu_{-3}=-\Sigma\frac{(x+1)(x+2)(x+4)}{2}\mu_x.$$

Similarly we find

$$\mu_{-2}=\Sigma\frac{(x+1)(x+3)(x+4)}{2}\mu_x$$

$$\mu_{-1}=-\Sigma\frac{(x+2)(x+3)(x+4)}{2.3}\mu_x.$$

But, as assumed in (g),

$$\mu_{-4}=\frac{(n+1)(n+2)(n+3)(n+4)}{2.3}\delta d_{-4}$$

$$\mu_{-3}=\frac{n(n+1)(n+2)(n+3)}{2.3}\delta d_{-3}$$

$$\mu_{-2}=\frac{(n-1)n(n+1)(n+2)}{2.3}\delta d_{-2}$$

$$\mu_{-1}=\frac{(n-2)(n-1)n(n+1)}{2.3}\delta d_{-1};$$

$$\left.\begin{aligned}\therefore\ \delta d_{-4}&=\ \ \ \ \Sigma\frac{(x+1)(x+2)(x+3)}{(n+1)(n+2)(n+3)(n+4)}\mu_x\\ \delta d_{-3}&=-3\Sigma\frac{(x+1)(x+2)(x+4)}{n(n+1)(n+2)(n+3)}\mu_x\\ \delta d_{-2}&=\ \ \ 3\Sigma\frac{(x+1)(x+3)(x+4)}{(n-1)n(n+1)(n+2)}\mu_x\\ \delta d_{-1}&=-\ \ \Sigma\frac{(x+2)(x+3)(x+4)}{(n-2)(n-1)n(n+1)}\mu_x\end{aligned}\right\}\ldots(k),$$

in which the summations under Σ are from $x=0$ to $x=n-4$ inclusive, and the complete form is embodied in each term.

By algebraically differencing back from the symbols v_0, a_0, β_0, γ_0, the algebraic values of the supplementary differences are found as follows:—

$$v_0\ \left|\begin{matrix}v_0\\ a_0\end{matrix}\right|\begin{matrix}v_0\\ -v_0+a_0\\ \beta_0\end{matrix}\left|\begin{matrix}v_0\\ -2v_0+a_0\\ v_0-a_0+\beta_0\\ \gamma_0\end{matrix}\right|\begin{matrix}v_0 & =\delta d_{-4}\\ -3v_0+a_0 & =\delta d_{-3}\\ 3v_0-2a_0+\beta_0 & =\delta d_{-2}\\ -\ v_0+a_0-\beta_0+\gamma_0 & =\delta d_{-1}\end{matrix}$$

$$\left.\begin{aligned}\therefore\ v_0&=\ \delta d_{-4}\\ a_0&=3\delta d_{-4}+\ \delta d_{-3}\\ \beta_0&=3\delta d_{-4}+2\delta d_{-3}+\delta d_{-2}\\ \gamma_0&=\ \delta d_{-4}+\ \delta d_{-3}+\delta d_{-2}+\ \delta d_{-1}\end{aligned}\right\}\ldots\ldots(l),$$

which, after substituting the values (k), will be found to be equivalent to the formulæ (h), before determined.

The generalization of the foregoing method, for any order (p) of differences, is perfectly analogous to what has been given, and the systematic law which obtains in the form of the final results is such that they may be readily put down for any other order of differences. It will be observed, that in the expressions for $\delta d_{-p}, \ldots \delta d_{-1}$, the numerical coefficients are the respective terms of the expansion of the binomial $(1-1)^{p-1}$; and that in each numerator the absent factor corresponds with the subnumeral indice of δd.

Thus, for sixth differences, we should have

$$v_m=\Sigma'\frac{(m-\overline{x+5})(m-\overline{x+4})(m-\overline{x+3})(m-\overline{x+2})(m-\overline{x+1})}{2.3.4.5}\delta f_x\ldots(\epsilon''),$$

$$\mu_x=\frac{(n-\overline{x+5})(n-\overline{x+4})(n-\overline{x+3})(n-\overline{x+2})(n-\overline{x+1})(n-x)}{2.3.4.5}\delta f_x\ldots(g'');$$

and, for the supplemental differences, we should obtain

$$\delta f_{-6}= \quad \Sigma\frac{(x+1)(x+2)(x+3)(x+4)(x+5)}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}\mu_x$$

$$\delta f_{-5}=-\ 5\Sigma\frac{(x+1)(x+2)(x+3)(x+4)(x+6)}{n(n+1)(n+2)(n+3)(n+4)(n+5)}\mu_x$$

$$\delta f_{-4}=\ 10\Sigma\frac{(x+1)(x+2)(x+3)(x+5)(x+6)}{(n-1)n(n+1)(n+2)(n+3)(n+4)}\mu_x$$

$$\delta f_{-3}=-10\Sigma\frac{(x+1)(x+2)(x+4)(x+5)(x+6)}{(n-2)(n-1)n(n+1)(n+2)(n+3)}\mu_x$$

$$\delta f_{-2}=\ 5\Sigma\frac{(x+1)(x+3)(x+4)(x+5)(x+6)}{(n-3)(n-2)(n-1)n(n+1)(n+2)}\mu_x$$

$$\delta f_{-1}=-\ \Sigma\frac{(x+2)(x+3)(x+4)(x+5)(x+6)}{(n-4)(n-3)(n-2)(n-1)n(n+1)}\mu_x \quad . . . \ (k'').$$

PRACTICAL ADAPTATION OF FOURTH DIFFERENCE FORMULÆ.

To simplify the formulæ for calculation, let $x'=(n-4)-x$, and tabulate the factorial functions

$$k=\frac{(x'+1)(x'+2)(x'+3)(x'+4)}{2.3.4},$$

$$p=+\frac{4k_x}{x+4},\ q=-\frac{12k_x}{x+3},\ r=+\frac{12k_x}{x+2},\ s=-\frac{4k_x}{x+1}.$$

Then we shall have

$$\left.\begin{aligned}\delta d_{-4}&=\Sigma\frac{kp}{k_n}\delta d_x\\ \delta d_{-3}&=\Sigma\frac{kq}{k_{n-1}}\delta d_x\\ \delta d_{-2}&=\Sigma\frac{kr}{k_{n-2}}\delta d_x\\ \delta d_{-1}&=\Sigma\frac{ks}{k_{n-3}}\delta d_x\end{aligned}\right\} \quad . . . . \ (n);$$

also,

$$v_m=p_m\delta d_{-4}+p_{m-1}\delta d_{-3}+p_{m-2}\delta d_{-2}\ . . . +\delta d_{m-4};$$

that is,

$$\begin{aligned}v_0&=\ \delta d_{-4}\\ v_1&=\ 4\delta d_{-4}+\ \delta d_{-3}\\ v_2&=10\delta d_{-4}+\ 4\delta d_{-3}+\ \delta d_{-2}\\ v_3&=20\delta d_{-4}+10\delta d_{-3}+\ 4\delta d_{-2}+\ \delta d_{-1}\\ v_4&=35\delta d_{-4}+20\delta d_{-3}+10\delta d_{-2}+4\delta d_{-1}+\delta d_0\\ &\&c. \qquad\qquad \&c.\end{aligned}$$

But it will be found that the several values of v may be otherwise readily deduced by successive summation of the columns of differences.

The tables here required are easily prepared, and are as follows:—

Subsidiary Tables for Fourth Differences.

x'	k
&c.	&c.
20	10626
19	8855
18	7315
17	5985
16	4845
15	3876
14	3060
13	2380
12	1820
11	1365
10	1001
9	715
8	495
7	330
6	210
5	126
4	70
3	35
2	15
1	5
0	1

x	p	q	r	s
0	+ 1	− 4	+ 6	− 4
1	4	15	20	10
2	10	36	45	20
3	20	70	84	35
4	35	120	140	56
5	56	189	216	84
6	84	280	315	120
7	120	396	440	165
8	165	540	594	220
9	220	715	780	286
10	286	924	1001	364
11	364	1170	1260	455
12	455	1456	1560	560
13	560	1785	1904	680
14	680	2160	2295	816
15	816	2584	2736	969
16	969	3060	2823	1140
&c.	&c.	&c.	&c.	&c.

These tables may be applied with the greatest facility to the resolution of any given case by means of the following simple rule, suggested by the formulæ (m):—

RULE.—*Beginning with $x'=n$, transcribe the several numbers to the end of Column (k). The first four of these will be the four divisors of the respective formulæ, and the remaining numbers will be the multipliers of each set. Opposite to these multipliers write down as many leading values of p, q, r, s, in four adjoining columns. Then the numerical coefficients of the formulæ expressing the four supplementary fourth differences will be found by taking the horizontal products kp, kq, kr, ks, and dividing the sets by the respective divisors before mentioned.*

Example 1.—Let there be only one fourth difference; then $n=4$, and there will be only one value of x, viz., $x=x'=0$.

The numbers from the tables will stand thus:—

k					
	70	... k_n			
	35				
	15				
	5	p	q	r	s
	1	+1	−4	+6	−4

$$\therefore\ \delta d_{-4}=+\frac{1}{70}\delta d_0 \qquad\qquad \delta d_{-2}=+\frac{6}{15}\delta d_0=+\frac{2}{5}\delta d_0$$

$$\delta d_{-3}=-\frac{4}{35}\delta d_0 \qquad\qquad \delta d_{-1}=-\frac{4}{5}\delta d_0.$$

Hence, multiplying by 70 to obviate fractions, the values of v are found by continued summation of each preceding column, as follows; the supplementary differences being cut off by horizontal lines:—

$70\delta d$	70γ	70β	$70a$	$70v$
$+\ \delta d_0$	$+\ \delta d_0$	$+\ \delta d_0$	$+\ \delta d_0$	$+\delta d_0$
$-\ 8\delta d_0$	$-\ 7\delta d_0$	$-\ 6\delta d_0$	$-\ 5\delta d_0$	$-4\delta d_0$
$+28\delta d_0$	$+21\delta d_0$	$+15\delta d_0$	$+10\delta d_0$	$+6\delta d_0$
$-56\delta d_0$	$-35\delta d_0$	$-20\delta d_0$	$-10\delta d_0$	$-4\delta d_0$
$+70\delta d_0$	$+35\delta d_0$	$+15\delta d_0$	$+\ 5\delta d_0$	$+\ \delta d_0$

$$\therefore\ v_0=+\frac{1}{70}\delta d_0 \qquad v_2=+\frac{6}{70}\delta d_0 \qquad v_4=\frac{1}{70}\delta d_0$$

$$v_1=-\frac{4}{70}\delta d_0 \qquad v_3=-\frac{4}{70}\delta d_0$$

Example 2.—Let there be two fourth differences, d_0, d_1; then $n=5$, and the tabular numbers are

	p	q	r	s
126				
70				
35				
15				
5	+1	− 4	+ 6	− 4
1	4	15	20	10

$$\therefore\ \delta d_{-4}=\frac{5}{126}\delta d_0+\frac{4}{126}\delta d_1 \qquad\qquad \delta d_{-2}=\frac{6}{7}\delta d_0+\frac{4}{7}\delta d_1$$

$$\delta d_{-3}=-\frac{4}{14}\delta d_0-\frac{3}{14}\delta d_1 \qquad\qquad \delta d_{-1}=-\frac{4}{3}\delta d_0-\frac{2}{3}\delta d_1$$

And, operating only with the numerical coefficients,

$126\delta d$	126γ	126β	$126a$	$126v$
+ 5+ 4	+ 5+ 4	+ 5+ 4	+ 5+ 4	+ 5+ 4
− 36−27	− 31−23	−26−19	−21−15	−16−11
+108+72	+ 77+49	+51+30	+30+15	+14+ 4
−168−84	− 91−35	−40− 5	−10+10	+ 4+14
126 0	+ 35−35	− 5−40	−15−30	−11−16
126	+ 35+91	+30+51	+15+21	+ 4+ 5

$$\therefore v_0 = \frac{5}{126}\delta d_0 + \frac{4}{126}\delta d_1 \qquad v_3 = \frac{4}{126}\delta d_0 + \frac{14}{126}\delta d_1$$

$$v_1 = -\frac{16}{126}\delta d_0 - \frac{11}{126}\delta d_1 \qquad v_4 = -\frac{11}{126}\delta d_0 - \frac{16}{126}\delta d_1$$

$$v_2 = \frac{14}{126}\delta d_0 + \frac{4}{126}\delta d_1 \qquad v_5 = \frac{4}{126}\delta d_0 + \frac{5}{126}\delta d_1$$

Example 3.—Let there be three fourth differences; then $n=6$, and

k				
210†				
126				
70				
35	p	q	r	s
15	+ 1	− 4	+ 6	− 4
5	4	15	20	10
1	10	36	45	20

$$\because \delta d_{-4} = \Sigma\frac{kp}{210} = \frac{3}{42}\delta d_0 + \frac{4}{42}\delta d_1 + \frac{2}{42}\delta d_2$$

$$\delta d_{-3} = \Sigma\frac{kq}{126} = -\frac{20}{42}\delta d_0 - \frac{25}{42}\delta d_1 - \frac{12}{42}\delta d_2$$

$$\delta d_{-2} = \Sigma\frac{kr}{70} = \frac{18}{14}\delta d_0 + \frac{20}{14}\delta d_1 + \frac{9}{14}\delta d_2$$

$$\delta d_{-1} = \Sigma\frac{ks}{35} = -\frac{12}{7}\delta d_0 - \frac{10}{7}\delta d_1 - \frac{4}{7}\delta d_2$$

And, by summation of the coefficients, as before,

$42\delta d$	42γ	42β	$42a$	$42v$
+ 3+ 4+ 2	+ 3+ 4+ 2	+ 3+ 4+ 2	+ 3+ 4+ 2	+3 +4+2
−20−25−12	−17−21−10	−14−17− 8	−11−13− 6	−8− 9−4
+54+60+27	+37+39+17	+23+22+ 9	+12+ 9+ 3	+4 * −1
−72−60−24	−35−21− 7	−12+ 1+ 2	* +10+ 5	+4+10+4
42 0 0	+ 7−21− 7	− 5−20− 5	− 5−10 *	−1 * +4
42 0	+ 7+21− 7	+ 2+ 1−12	− 3− 9−12	−4− 9−8
42	+ 7+21+35	+ 9+22+23	+ 6+13+11	+2+ 4+3

$$\therefore v_0 = \frac{3}{42}\delta d_0 + \frac{4}{42}\delta d_1 + \frac{2}{42}\delta d_2$$

$$v_1 = -\frac{8}{42}\delta d_0 - \frac{9}{42}\delta d_1 - \frac{4}{42}\delta d_2$$

† The numbers k put on a separate slip may be mechanically placed in position upon the other table, and thus all transcription will be dispensed with.

$$v_2= \quad \frac{4}{42}\delta d_0 \quad * \quad -\frac{1}{42}\delta d_2$$

$$v_3= \quad \frac{4}{42}\delta d_0+\frac{10}{42}\delta d_1+\frac{4}{42}\delta d_2$$

$$v_4=-\frac{1}{42}\delta d_0 \quad * \quad +\frac{4}{42}\delta d_2$$

$$v_5=-\frac{4}{42}\delta d_0-\frac{9}{42}\delta d_1-\frac{8}{42}\delta d_2$$

$$v_6= \quad \frac{2}{42}\delta d_0+\frac{4}{42}\delta d_1+\frac{3}{42}\delta d_2$$

The accuracy of the work will in all cases be checked by the symmetry of the coefficients, as well as by the condition $\Sigma v=0$.

Similarly for four fourth differences, or $n=7$, the results are

$$\delta d_{-4}= \quad \frac{7}{66}\delta d_0+\frac{12}{66}\delta d_1+\frac{10}{66}\delta d_2+\frac{4}{66}\delta d_3$$

$$\delta d_{-3}=-\frac{28}{42}\delta d_0-\frac{45}{42}\delta d_1-\frac{36}{42}\delta d_2-\frac{14}{42}\delta d_3$$

$$\delta d_{-2}= \quad \frac{70}{42}\delta d_0+\frac{100}{42}\delta d_1+\frac{75}{42}\delta d_2+\frac{28}{42}\delta d_3$$

$$\delta d_{-1}=-\frac{28}{14}\delta d_0-\frac{30}{14}\delta d_1-\frac{20}{14}\delta d_2-\frac{7}{14}\delta d_3;$$

and, by summation of the coefficients as before,

$$v_0= \quad \frac{7}{66}\delta d_0+\frac{12}{66}\delta d_1+\frac{10}{66}\delta d_2+\frac{4}{66}\delta d_3$$

$$v_1=-\frac{16}{66}\delta d_0-\frac{159}{7.66}\delta d_1-\frac{116}{7.66}\delta d_2-\frac{6}{66}\delta d_3$$

$$v_2= \quad \frac{4}{66}\delta d_0-\frac{40}{7.66}\delta d_1-\frac{59}{7.66}\delta d_2-\frac{4}{66}\delta d_3$$

$$v_3= \quad \frac{8}{66}\delta d_0+\frac{20}{66}\delta d_1+\frac{80}{7.66}\delta d_2+\frac{3}{66}\delta d_3$$

$$v_4= \quad \frac{3}{66}\delta d_0+\frac{80}{7.66}\delta d_1+\frac{20}{66}\delta d_2+\frac{8}{66}\delta d_3$$

$$v_5=-\frac{4}{66}\delta d_0-\frac{59}{7.66}\delta d_1-\frac{40}{7{\cdot}66}\delta d_2+\frac{4}{66}\delta d_3$$

$$v_6=-\frac{6}{66}\delta d_0-\frac{116}{7.66}\delta d_1-\frac{159}{7.66}\delta d_2-\frac{16}{66}\delta d_3$$

$$v_7= \quad \frac{4}{66}\delta d_0+\frac{10}{66}\delta d_1+\frac{12}{66}\delta d_2+\frac{7}{66}\delta d_3.$$

Again, for five fourth differences,

$$\delta d_{-4}= \frac{14}{99}\delta d_0+\frac{28}{99}\delta d_1+\frac{30}{99}\delta d_2+\frac{20}{99}\delta d_3+\frac{7}{99}\delta d_4$$

$$\delta d_{-3}=-\frac{28}{33}\delta d_0-\frac{35}{22}\delta d_1-\frac{18}{11}\delta d_2-\frac{35}{33}\delta d_3-\frac{4}{11}\delta d_4$$

$$\delta d_{-2}= 2\delta d_0+\frac{10}{3}\delta d_1+\frac{45}{14}\delta d_2+2\delta d_3+\frac{2}{3}\delta d_4$$

$$\delta d_{-1}=-\frac{20}{9}\delta d_0-\frac{25}{9}\delta d_1-\frac{50}{21}\delta d_2-\frac{25}{18}\delta d_3-\frac{4}{9}\delta d_4;$$

$$v_0= \frac{14}{99}\delta d_0+\frac{28}{99}\delta d_1+\frac{30}{99}\delta d_2+\frac{20}{99}\delta d_3+\frac{7}{99}\delta d_4$$

$$v_1=-\frac{28}{99}\delta d_0-\frac{91}{198}\delta d_1-\frac{42}{99}\delta d_2-\frac{25}{99}\delta d_3-\frac{8}{99}\delta d_4$$

$$v_2= \frac{2}{99}\delta d_0-\frac{20}{99}\delta d_1-\frac{139}{14.33}\delta d_2-\frac{22}{99}\delta d_3-\frac{8}{99}\delta d_4$$

$$v_3= \frac{12}{99}\delta d_0+\frac{30}{99}\delta d_1+\frac{40}{7.33}\delta d_2+\frac{3}{66}\delta d_3 \quad *$$

$$v_4= \frac{9}{99}\delta d_0+\frac{30}{99}\delta d_1+\frac{115}{7.33}\delta d_2+\frac{30}{99}\delta d_3+\frac{9}{99}\delta d_4$$

$$v_5= * \quad +\frac{3}{66}\delta d_1+\frac{40}{7.33}\delta d_2+\frac{30}{99}\delta d_3+\frac{12}{99}\delta d_4$$

$$v_6=-\frac{8}{99}\delta d_0-\frac{22}{99}\delta d_1-\frac{139}{14.33}\delta d_2-\frac{20}{99}\delta d_3+\frac{2}{99}\delta d_4$$

$$v_7=-\frac{8}{99}\delta d_0-\frac{25}{99}\delta d_1-\frac{42}{99}\delta d_2-\frac{91}{198}\delta d_3-\frac{28}{99}\delta d_4$$

$$v_8= \frac{7}{99}\delta d_0+\frac{20}{99}\delta d_1+\frac{30}{99}\delta d_2+\frac{28}{99}\delta d_3+\frac{14}{99}\delta d_4.$$

The formulæ may in like manner be constructed for any proposed number of differences. And, finally, to adjust the given original values V_0, V_1, V_2, &c., the calculated corrections or adjustments v_0, v_1, v_2, &c., are to be respectively applied according to their several algebraic signs.

We have yet to make a further most important improvement and extension of the method, by the introduction of certain considerations, connected with the theory of the differences of algebraic functions, which have the effect of concentrating the general principles of the investigation within the narrowest possible limits, and of finally bringing the subject under a remarkably comprehensive point of view.

As the coefficients of δd_x, in the equations (θ), involve x algebraically to powers not exceeding the seventh, it follows that the

eighth difference of any nine successive values of a set of each of these coefficients will be zero. Hence it will appear that these four equations will all of them be fulfilled if the several differences be replaced by any of the following horizontal sets of numerical values:—

δd_{-4}	δd_{-3}	δd_{-2}	δd_{-1}	δd_0	δd_1	δd_2	δd_3	δd_4	δd_5	δd_6	δd_7	
+1	−8	+28	−56	+70	−56	+28	−8	+1	0	0	0	&c.
0	+1	−8	+28	−56	+70	−56	+28	−8	+1	0	0	&c.
0	0	+1	−8	+28	−56	+70	−56	+28	−8	+1	0	&c.
0	0	0	+1	−8	+28	−56	+70	−56	+28	−8	+1	&c.
		&c.				&c.			&c.			

for each formula will then, in fact, express the eighth difference of nine consecutive coefficients; and, for the reason just stated, each of these eighth differences of specified algebraic functions must necessarily vanish. Therefore, as the same must hold with any positive or negative multiple of a horizontal set of numbers, it will follow generally that all the requisite conditions (θ) will be satisfied provided that

$$\left.\begin{array}{l} \delta d_{-4} = \phi_0 \\ \delta d_{-3} = -8\phi_0 + \phi_1 \\ \delta d_{-2} = +28\phi_0 - 8\phi_1 + \phi_2 \\ \delta d_{-1} = -56\phi_0 + 28\phi_1 - 8\phi_2 + \phi_3 \end{array}\right\} \quad \ldots \quad (1)$$

$$\left.\begin{array}{l} \delta d_0 = +70\phi_0 - 56\phi_1 + 28\phi_2 - 8\phi_3 \\ \delta d_1 = -56\phi_0 + 70\phi_1 - 56\phi_2 + 28\phi_3 \\ \delta d_2 = +28\phi_0 - 56\phi_1 + 70\phi_2 - 56\phi_3 \\ \delta d_3 = -8\phi_0 + 28\phi_1 - 56\phi_2 + 70\phi_3 \end{array}\right\} \quad \ldots \quad (2)$$

$$\left.\begin{array}{l} \delta d_4 = \phi_0 - 8\phi_1 + 28\phi_2 - 56\phi_3 \\ \delta d_5 = \ldots\ldots \phi_1 - 8\phi_2 + 28\phi_3 \\ \delta d_6 = \ldots\ldots\ldots \phi_2 - 8\phi_3 \\ \delta d_7 = \ldots\ldots\ldots\ldots \phi_3 \end{array}\right\} \quad \ldots \quad (3)$$

where ϕ_0, ϕ_1, ϕ_2, ϕ_3, represent a set of arbitrary or determinate quantities, which may evidently, in like manner, be extended to any number. These algebraic expressions are moreover the fully developed eighth differences of the series

$$0,\ \phi_0,\ \phi_1,\ \phi_2,\ \phi_3,\ 0;$$

and hence also the original quantities v_0, v_1, v_2, v_3, v_4, &c., are the fourth differences of the same series, and are therefore expressed as follows:—

$$
\begin{aligned}
v_0 &= \phi_0 \\
v_1 &= -4\phi_0 + \phi_1 \\
v_2 &= +6\phi_0 - 4\phi_1 + \phi_2 \\
v_3 &= -4\phi_0 + 6\phi_1 - 4\phi_2 + \phi_3 \\
v_4 &= \phi_0 - 4\phi_1 + 6\phi_2 - 4\phi_3 \\
v_5 &= \ldots\ldots \phi_1 - 4\phi_2 + 6\phi_3 \\
v_6 &= \ldots\ldots\ldots \phi_2 - 4\phi_3 \\
v_7 &= \ldots\ldots\ldots\ldots \phi_3 \quad \ldots\ldots (4).
\end{aligned}
$$

The general equations, arising out of the principle of least squares, are thus effectually embodied in the simple condition that the required series of corrections (v) shall be the expanded fourth differences of an interior and more concentrated series of values (ϕ), four less in number, or the same in number as that of the fourth differences d_0, d_1, d_2, &c.

This view of the subject may be arrived at in a still more direct and simple manner by a consideration of the initial equations (e) in connexion with the fundamental formula (ϵ).

Suppose four inner sets of differences, denoted by w, y, z, ϕ, to be inserted in the scheme of adjustments, and to be determined from v_0, v_1, v_2, v_3, &c., by successive summation, thus:—

$$
\begin{array}{rllll|c|cccc}
 & & & & & & & & & \delta d_{-4} \\
 & & & & & & & \beta_{-2} & \gamma_{-3} & \delta d_{-3} \\
 & & & & & v_0 & \alpha_{-1} & \beta_{-1} & \gamma_{-2} & \delta d_{-2} \\
 & & & & w_0 & v_1 & \alpha_0 & \beta_0 & \gamma_{-1} & \delta d_{-1} \\
 & & & y_0 & w_1 & v_2 & \alpha_1 & \beta_1 & \gamma_0 & \delta d_0 \\
 & & z_0 & y_1 & w_2 & v_3 & \alpha_2 & \beta_2 & \gamma_1 & \delta d_1 \\
\text{Foci} & \left\{\begin{array}{l}\phi_0 \\ \phi_1 \\ \vdots \\ \phi_{n-4}\end{array}\right. & \begin{array}{l} z_1 \\ z_2 \\ \vdots \\ z_{n-3}\end{array} & \begin{array}{l} y_2 \\ y_3 \\ \vdots \\ y_{n-2}\end{array} & \begin{array}{l} w_3 \\ w_4 \\ \vdots \\ w_{n-1}\end{array} & \begin{array}{l} v_4 \\ v_5 \\ \vdots \\ v_n\end{array} & \begin{array}{l} \alpha_3 \\ \alpha_4 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n\end{array} & \begin{array}{l} \beta_3 \\ \vdots \\ \beta_{n-2} \\ \beta_{n-1} \\ \beta_n\end{array} & \begin{array}{l} \gamma_2 \\ \vdots \\ \gamma_{n-3} \\ \gamma_{n-2} \\ \gamma_{n-1} \\ \gamma_n\end{array} & \begin{array}{l} \vdots \\ \delta d_{n-4} \\ \delta d_{n-3} \\ \delta d_{n-2} \\ \delta d_{n-1} \\ \delta d_n\end{array}
\end{array}
$$

and by formulæ analagous to (ϵ) we shall have

$$
\begin{aligned}
w_n &= \Sigma v \\
y_{n-1} &= \Sigma w = \Sigma(n+1-x)v_x \\
z_{n-2} &= \Sigma y = \Sigma \frac{(n+1-x)(n+2-x)}{2} v_x \\
\phi_{n-3} &= \Sigma z = \Sigma \frac{(n+1-x)(n+2-x)(n+3-x)}{2.3} v_x.
\end{aligned}
$$

But, by equations (*e*), $\Sigma v = \Sigma xv = \Sigma x^2 v = \Sigma x^3 v = 0$;

$$\therefore w_n = y_{n-1} = z_{n-2} = \phi_{n-3} = 0.$$

Hence these inner sets of differences will severally terminate with the values w_{n-1}, y_{n-2}, z_{n-3}, ϕ_{n-4}, as they appear in the scheme. If, however, any more inner sets of differences were to be constructed, it is evident that, unless $\Sigma x^4 v = 0$, the number of terms, instead of continuing to contract, would become indefinitely prolonged. In consequence of this remarkable peculiarity, which is evidently applicable to any order of differences, the series of values of ϕ are here designated the *focal adjustments* with respect to the differences under consideration.

Whatever be the values of ϕ_0, ϕ_1, ϕ_2, ϕ_3, &c., which form the radical basis of the differencing scheme, Σv^2 will necessarily be the least possible out of all sets of quantities which can have for their fourth differences the series δd_0, δd_1, δd_2, &c., or the more extended series δd_{-4}, δd_{-3}, δd_{-2}, &c. Therefore, as this condition places no restriction upon the values of ϕ_0, ϕ_1, ϕ_2, ϕ_3, &c., they can be made to fulfil any other condition that may be required. The other sets of differences being linear functions of ϕ_0, ϕ_1, ϕ_2, ϕ_3, &c., and exceeding them in number, are, however, not wholly independent, and it is evident that only as many conditions as focal corrections are admissible.

When, for example, the requisite adjustments of four fourth differences are given, the corresponding focal adjustments are determined by the four equations (2), viz.,

$$\left.\begin{aligned} 70\phi_0 - 56\phi_1 + 28\phi_2 - \ 8\phi_3 &= \delta d_0 \\ -56\phi_0 + 70\phi_1 - 56\phi_2 + 28\phi_3 &= \delta d_1 \\ 28\phi_0 - 56\phi_1 + 70\phi_2 - 56\phi_3 &= \delta d_2 \\ -\ 8\phi_0 + 28\phi_1 - 56\phi_2 + 70\phi_3 &= \delta d_3 \end{aligned}\right\} \quad \ldots\ldots (5);$$

which give

$$\left.\begin{aligned} 462\phi_0 &= 49\delta d_0 + \ 84\delta d_1 + \ 70\delta d_2 + 28\delta d_3 \\ 462\phi_1 &= 84\delta d_0 + 177\delta d_1 + 164\delta d_2 + 70\delta d_3 \\ 462\phi_2 &= 70\delta d_0 + 164\delta d_1 + 177\delta d_2 + 84\delta d_3 \\ 462\phi_3 &= 28\delta d_0 + \ 70\delta d_1 + \ 84\delta d_2 + 49\delta d_3 \end{aligned}\right\} \quad \ldots\ldots (p).$$

The preceding and following supplementary differences are likewise found by substitution in (1) and (3), and the results will be found to accord with those of the same example on page 157. Hence also the focal values may be otherwise found by a continuation of the process of summation, with respect to the coefficients, thus :—

462v	462w	462y
49 + 84 + 70 + 28	49 + 84 + 70 + 28	49 + 84 + 70 + 28
− 112 − 159 − 116 − 42	− 63 − 75 − 46 − 14	− 14 + 9 + 24 + 14
28 − 40 − 59 − 28	− 35 − 115 − 105 − 42	− 49 − 106 − 81 − 28
56 + 140 + 80 + 21	21 + 25 − 25 − 21	− 28 − 81 − 106 − 49
21 + 80 + 140 + 56	42 + 105 + 115 + 35	14 + 24 + 9 − 14
− 28 − 59 − 40 + 28	14 + 46 + 75 + 63	+ 28 + 70 + 84 + 49
− 42 − 116 − 159 − 112	− 28 − 70 − 84 − 49	*
28 + 70 + 84 + 49	*	

462z	462φ
49 + 84 + 70 + 28	49 + 84 + 70 + 28
35 + 93 + 94 + 42	84 + 177 + 164 + 70
− 14 − 13 + 13 + 14	70 + 164 + 177 + 84
− 42 − 94 − 93 − 35	28 + 70 + 84 + 49
− 28 − 70 − 84 − 49	*
*	Focal coefficients.

Here the last set of numbers are the coefficients of the formulæ (p), and the symmetry of the numbers throughout is an effectual test of the accuracy of the work.

The following series of tables, deduced from equations analagous to (5), exhibit the similar coefficients for any number of differences and foci up to seven. As they are all positive, the algebraic sign is not required.

Tables A.—*Coefficients for deducing Focal Values from their Eighth Differences or from assumed Fourth Difference Adjustments.*

70φ

1

126φ

5	4
4	5

42φ

3	4	2
4	7	4
2	4	3

462φ

49	84	70	28
84	177	164	70
70	164	177	84
28	70	84	49

198φ

28	56	60	40	14
56	133	156	110	40
60	156	$204\frac{3}{7}$	156	60
40	110	156	133	56
14	40	60	56	28

4290φ

756	1680	2100	1800	1050	336
1680	4340	5880	5300	3200	1050
2100	5880	8715	8380	5300	1800
1800	5300	8380	8715	5880	2100
1050	3200	5300	5880	4340	1680
336	1050	1800	2100	1680	756

858ϕ

180	432	600	600	450	240	72
432	1188	1776	1860	1440	786	240
600	1776	2868	3176	2560	1440	450
600	1860	3176	3743	3176	1860	600
450	1440	2560	3176	2868	1776	600
240	786	1440	1860	1776	1188	432
72	240	450	600	600	432	180

The darker lines in each table show how symmetrically certain coefficients distribute themselves within the four sides of each concentric rectangle. Also the several sets of coefficients are precisely the same whether they are read off vertically or horizontally. The symmetrical relations thus exhibited may indeed be inferred from a like symmetry in the coefficients of the original equations from which the tabular numbers are derived; and they present an efficient and useful check on the accuracy of the calculation. An improved method of constructing these tables will be given hereafter. For more than seven values the resulting fractions are so cumbrous that it would be more convenient to tabulate them in decimals.

If the necessary adjustments of a number of fourth differences can be first assigned, the focal adjustments may be calculated with these tabular coefficients according to formulæ similar to (p), and thence the adjustments of the primitive quantities may be found either by successive differencing or by substitution in expressions of the same kind as (4).

It will be observed that the foregoing methods of calculation proceed upon an assumed previous knowledge of the adjustments required to be made in the fourth differences, and that, in rectifying the progression, these must again depend in some measure upon the judgment and discretion of the computer. In order, however, to determine theoretically the values of δd_0, δd_1, δd_2, &c., so as to effect the requisite adjustment of the quantities V with the least possible displacements, we may assume as an available new condition that the sum of the squares of all the amended fourth differences shall also be a minimum. Resuming the case of four foci, which will sufficiently indicate the general principles of the method which follows from this hypothesis, let the differences be carried out more extensively as follows:—

$$
\begin{array}{cccc|c|ccc|c|ccc|c|cccc}
 & & & & & & & & \delta d_{-4} & & & & & & & & \\
 & & & & & & & \gamma_{-3} & & \delta e_{-3} & & & & & & & \\
 & & & & & & \beta_{-2} & & \delta d_{-3} & & \delta f_{-2} & & & & & & \\
 & & & & & \alpha_{-1} & & \gamma_{-2} & & \delta e_{-2} & & \delta g_{-1} & & & & & \\
 & & & & v_0 & & \beta_{-1} & & \delta d_{-2} & & \delta f_{-1} & & \delta h_0 & & & & \\
 & & & w_0 & & \alpha_0 & & \gamma_{-1} & & \delta e_{-1} & & \delta g_0 & & \delta i_0 & & & \\
 & & y_0 & & v_1 & & \beta_0 & & \delta d_{-1} & & \delta f_0 & & \delta h_1 & & \delta k_0 & & \\
 & z_0 & & w_1 & & \alpha_1 & & \gamma_0 & & \delta e_0 & & \delta g_1 & & \delta i_1 & & \delta l_0 & \\
\phi_0 & & y_1 & & v_2 & & \beta_1 & & \delta d_0 & & \delta f_1 & & \delta h_2 & & \delta k_1 & & \delta m_0 \\
 & z_1 & & w_2 & & \alpha_2 & & \gamma_1 & & \delta e_1 & & \delta g_2 & & \delta i_2 & & \delta l_1 & \\
\phi_1 & & y_2 & & v_3 & & \beta_2 & & \delta d_1 & & \delta f_2 & & \delta h_3 & & \delta k_2 & & \delta m_1 \\
 & z_2 & & w_3 & & \alpha_3 & & \gamma_2 & & \delta e_2 & & \delta g_3 & & \delta i_3 & & \delta l_2 & \\
\phi_2 & & y_3 & & v_4 & & \beta_3 & & \delta d_2 & & \delta f_3 & & \delta h_4 & & \delta k_3 & & \delta m_2 \\
 & z_3 & & w_4 & & \alpha_4 & & \gamma_3 & & \delta e_3 & & \delta g_4 & & \delta i_4 & & \delta l_3 & \\
\phi_3 & & y_4 & & v_5 & & \beta_4 & & \delta d_3 & & \delta f_4 & & \delta h_5 & & \delta k_4 & & \delta m_3 \\
 & z_4 & & w_5 & & \alpha_5 & & \gamma_4 & & \delta e_4 & & \delta g_5 & & \delta i_5 & & \delta l_4 & \\
 & & y_5 & & v_6 & & \beta_5 & & \delta d_4 & & \delta f_5 & & \delta h_6 & & \delta k_5 & & \\
 & & & w_6 & & \alpha_6 & & \gamma_5 & & \delta e_5 & & \delta g_6 & & \delta i_6 & & & \\
 & & & & v_7 & & \beta_6 & & \delta d_5 & & \delta f_6 & & \delta h_7 & & & & \\
 & & & & & \alpha_7 & & \gamma_6 & & \delta e_6 & & \delta g_7 & & & & & \\
 & & & & & & \beta_7 & & \delta d_6 & & \delta f_7 & & & & & & \\
 & & & & & & & \gamma_7 & & \delta e_7 & & & & & & & \\
 & & & & & & & & \delta d_7 & & & & & & & &
\end{array}
$$

And let the differences of V_0, V_1, V_2, V_3, &c., be extended in like manner and with an analogous notation, only omitting the symbol δ. Then the complete list of the adjusted fourth differences, expressed in terms of the focal quantities ϕ_0, ϕ_1, ϕ_2, ϕ_3, will be

$$
\begin{array}{l}
d_{-4}+\delta d_{-4}=d_{-4}+\phi_0 \\
d_{-3}+\delta d_{-3}=d_{-3}-8\phi_0+\phi_1 \\
d_{-2}+\delta d_{-2}=d_{-2}+28\phi_0-8\phi_1+\phi_2 \\
d_{-1}+\delta d_{-1}=d_{-1}-56\phi_0+28\phi_1-8\phi_2+\phi_3 \\
\hline
d_0+\delta d_0=d_0+70\phi_0-56\phi_1+28\phi_2-8\phi_3 \\
d_1+\delta d_1=d_1-56\phi_0+70\phi_1-56\phi_2+28\phi_3 \\
d_2+\delta d_2=d_2+28\phi_0-56\phi_1+70\phi_2-56\phi_3 \\
d_3+\delta d_3=d_3-8\phi_0+28\phi_1-56\phi_2+70\phi_3 \\
\hline
d_4+\delta d_4=d_4+\phi_0-8\phi_1+28\phi_2-56\phi_3 \\
d_5+\delta d_5=d_5 \ldots\ldots +\phi_1-8\phi_2+28\phi_3 \\
d_6+\delta d_6=d_6 \ldots\ldots\ldots +\phi_2-8\phi_3 \\
d_7+\delta d_7=d_7 \ldots\ldots\ldots\ldots +\phi_3
\end{array}
$$

The adjustments under consideration are specifically those of the fourth differences d_0, d_1, d_2, d_3; but the four supplementary differences which precede, and also those which follow, are included in this list, in order to avoid any material disturbance of these adjoining differences, by operating upon the results which include the whole of the effects developed by the focal adjustments.

Now, in order to determine the focal quantities ϕ_0, ϕ_1, ϕ_2, &c., when the sum of the squares of these adjusted differences is a minimum, the resolving equations, according to the rule on page 145, will be found by first multiplying each equation by the coefficient of ϕ_0, and equating the sum of the products with zero; and then

severally proceeding to operate in the same manner with respect to ϕ_1, ϕ_2, ϕ_3. Hence, observing that

$$d_{-4}-8d_{-3}+28d_{-2}-56d_{-1}+70d_0-56d_1+28d_2-8d_3+d_4$$

is the eighth difference of the series

$$d_{-4},\ d_{-3},\ d_{-2},\ d_{-1},\ d_0,\ d_1,\ d_2,\ d_3,\ d_4,$$

and therefore equal to m_0; and that the other similar sets of products reduce in exactly the same manner, the required equations of condition are simply

$$\left.\begin{array}{l} m_0+\delta m_0=0 \\ m_1+\delta m_1=0 \\ m_2+\delta m_2=0 \\ m_3+\delta m_3=0 \end{array}\right\} \text{ or } \left\{\begin{array}{l} \delta m_0=-m_0 \\ \delta m_1=-m_1 \\ \delta m_2=-m_2 \\ \delta m_3=-m_3 \end{array}\right\} \quad . \ . \ . \ (w)$$

These show that the twelfth differences of V_0, V_1, V_2, &c., must be neutralized by the sixteenth differences of ϕ_0, ϕ_1, ϕ_2, &c.

The sixteenth differences, viz., δm_0, δm_1, δm_2, δm_3, being thus known, the values of ϕ_0, ϕ_1, ϕ_2, ϕ_3, are, according to the table of coefficients, page 311, vol. xi., determined by the equations

$$\left.\begin{array}{r} 12870\phi_0-11440\phi_1+\ 8008\phi_2-\ 4368\phi_3=\delta m_0 \\ -11440\phi_0+12870\phi_1-11440\phi_2+\ 8008\phi_3=\delta m_1 \\ 8008\phi_0-11440\phi_1+12870\phi_2-11440\phi_3=\delta m_2 \\ -4368\phi_0+\ 8008\phi_1-11440\phi_2+12870\phi_3=\delta m_3 \end{array}\right\} \quad . \ . \ (6)$$

By solving these equations we get the following formulæ for computation with four foci:—

$$\left.\begin{array}{l} 277134\phi_0=\ 605\delta m_0+1320\delta m_1+1188\delta m_2+\ 440\delta m_3 \\ 277134\phi_1=1320\delta m_0+3165\delta m_1+3048\delta m_2+1188\delta m_3 \\ 277134\phi_2=1188\delta m_0+3048\delta m_1+3165\delta m_2+1320\delta m_3 \\ 277134\phi_3=\ 440\delta m_0+1188\delta m_1+1320\delta m_2+\ 605\delta m_3 \end{array}\right\} \quad . \ (9)$$

The determination of these coefficients is indeed evidently the same as would be obtained by the former method when applied to eighth instead of fourth differences; and they are here similarly tabulated up to seven differences, or seven foci.

TABLES B.—*Coefficients for deducing Focal Adjustments from Twelfth Differences.*

$12870.\phi$

1

$24310.\phi$

9	8
8	9

$24310.\phi$

25	40	20
40	73	40
20	40	25

$277134.\phi$

605	1320	1188	440
1320	3165	3048	1188
1188	3048	3165	1320
440	1188	1320	605

8398.ϕ

33	88	108	72	22
88	253	328	228	72
108	328	$449\frac{4}{11}$	328	108
72	228	328	253	88
22	72	108	88	33

293930.ϕ

1859	5720	8580	7800	4290	1144
5720	18755	29480	27780	15720	4290
8580	29480	48455	47480	27780	7800
7800	27780	47480	48455	29480	8580
4290	15720	27780	29480	18755	5720
1144	4290	7800	8580	5720	1859

125970.ϕ

1183	4056	7020	7800	5850	2808	676
4056	14703	26520	30420	23400	11466	2808
7020	26520	49695	58920	46620	23400	5850
7800	30420	58920	72195	58920	30420	7800
5850	23400	46620	58920	49695	26520	7020
2808	11466	23400	30420	26520	14703	4056
676	2808	2850	7800	7020	4056	1183

The coefficients of each higher table may, with a considerable saving of calculation, be deduced from those of the table immediately preceding it by the following method:—

Distinguishing by accentuation the symbols which appertain to the preceding table, let the equations analogous to (6) be

$$\left.\begin{array}{l} A_0\phi'_0-A_1\phi'_1+A_2\phi'_2\ldots\mp A_{n-1}\phi'_{n-1}=\delta m_0 \\ -A_1\phi'_0+A_0\phi'_1-A_1\phi'_2\ldots\pm A_{n-2}\phi'_{n-1}=\delta m_1 \\ A_2\phi'_0-A_1\phi'_1+A_0\phi'_2\ldots\mp A_{n-3}\phi'_{n-1}=\delta m_2 \\ \vdots \\ \mp A_{n-1}\phi'_0\pm A_{n-2}\phi'_1\mp A_{n-3}\phi'_2\ldots+A_0\phi'_{n-1}=\delta m_{n-1} \end{array}\right\}\ldots(7)$$

where A_0, A_1, A_2, &c., are factorial functions of n; and let the solution of these equations, as given by the tabular coefficients, be denoted thus:—

$$\left.\begin{array}{llllll}
\phi'_0 & =\phi'_{0,0}\delta m_0 & +\phi'_{0,1}\delta m_1 & +\phi'_{0,2}\delta m_2 \ldots & +\phi'_{0,n-1}\delta m_{n-1} \\
\phi'_1 & =\phi'_{1,0}\delta m_0 & +\phi'_{1,1}\delta m_1 & +\phi'_{1,2}\delta m_2 \ldots & +\phi'_{1,n-1}\delta m_{n-1} \\
\phi'_2 & =\phi'_{2,0}\delta m_0 & +\phi'_{2,1}\delta m_1 & +\phi'_{2,2}\delta m_2 \ldots & +\phi'_{2,n-1}\delta m_{n-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\phi'_{n-1} & =\phi'_{n-1,0}\delta m_0 & +\phi'_{n-1,1}\delta m_1 & +\phi'_{n-1,2}\delta m_2 \ldots & +\phi'_{n-1,n-1}\delta m_{n-1}
\end{array}\right\} \ldots (8)$$

Let the equations of condition which have to determine the coefficients of the sought table be

$$\left.\begin{array}{r}
A_0\phi_0-A_1\phi_1+A_2\phi_2 \ldots \mp A_{n-1}\phi_{n-1}\pm A_n\phi_n =\delta m_0 \\
-A_1\phi_0+A_0\phi_1-A_1\phi_2 \ldots \pm A_{n-2}\phi_{n-1}\mp A_{n-1}\phi_n=\delta m_1 \\
A_2\phi_0-A_1\phi_1+A_0\phi_2 \ldots \mp A_{n-3}\phi_{n-1}\pm A_{n-2}\phi_n=\delta m_2 \\
\mp A_{n-1}\phi_0\pm A_{n-2}\phi_1\mp A_{n-3}\phi_2 \ldots +A_0\phi_{n-1}-A_1\phi_n=\delta m_{n-1} \\
\pm A_n\phi_0\mp A_{n-1}\phi_1\pm A_{n-2}\phi_2 \ldots -A_1\phi_{n-1}+A_0\phi_n=\delta m_n
\end{array}\right\} \ldots (9)$$

Now, if $\delta m_0\mp A_n\phi_n$, $\delta m_1\pm A_{n-1}\phi_n$, $\delta m_2\mp A_{n-2}\phi_n$, ... $\delta m_{n-1}+A_1\phi_n$ be substituted in place of δm_0, δm_1, δm_2, ... δm_{n-1} respectively in the equation (7), they will be identical in form with the equations (9), omitting for the present the last one. Hence, by a like substitution in the first of the formulæ (8), we get

$$\begin{aligned}
\phi_0 &=\phi'_{0,0}(\delta m_0\mp A_n\phi_n)+\phi'_{0,1}(\delta m_1\pm A_{n-1}\phi_n) \\
&\quad +\phi'_{0,2}(\delta m_2\mp A_{n-2}\phi_n) \ldots +\phi'_{0,n-1}(\delta m_{n-1}+A_1\phi_n) \\
&=\phi'_0+(A_1\phi'_{0,n-1}-A_2\phi'_{0,n-2}+A_3\phi'_{0,n-3}-\&c.)\phi_n.
\end{aligned}$$

And in this way the formulæ (8) severally give

$$\left.\begin{array}{lll}
\phi_0 & =\phi'_0 & +\lambda_0\phi_n \\
\phi_1 & =\phi'_1 & +\lambda_1\phi_n \\
\phi_2 & =\phi'_2 & +\lambda_2\phi_n \\
\vdots & \vdots & \vdots \\
\phi_{n-1} & =\phi'_{n-1} & +\lambda_{n-1}\phi_n
\end{array}\right\} \ldots (10)$$

where

$$\left.\begin{array}{lllll}
\lambda_0 & =A_1\phi'_{0,n-1} & -A_2\phi'_{0,n-2} & +A_3\phi'_{0,n-3} & -\&c. \\
\lambda_1 & =A_1\phi'_{1,n-1} & -A_2\phi'_{1,n-2} & +A_3\phi'_{1,n-3} & -\&c. \\
\lambda_2 & =A_1\phi'_{2,n-1} & -A_2\phi'_{2,n-2} & +A_3\phi'_{2,n-3} & -\&c. \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\lambda_{n-1} & =A_1\phi'_{n-1,n-1} & -A_2\phi'_{n-1,n-2} & +A_3\phi'_{n-1,n-3} & -\&c.
\end{array}\right\} \ldots (11)$$

$$=A_1\phi'_{0,0}-A_2\phi'_{0,1}+A_3\phi'_{0,2}-\&c.$$

The last of the equations (9) now gives

$$\pm A_n(\phi'_0+\lambda_0\phi_n)\mp A_{n-1}(\phi'_1+\lambda_1\phi_n)\pm A_{n-2}(\phi'_2+\lambda_2\phi_n)$$
$$\ldots\ldots -A_1(\phi'_{n-1}+\lambda_{n-1}\phi_n)+A_0\phi_n=\delta m_n.$$

But

$$\pm A_n\phi'_0\mp A_{n-1}\phi'_1\pm A_{n-2}\phi'_2\ldots\ldots -A_1\phi'_{n-1}$$
$$=-A_1(\phi'_{n-1,0}\delta m_0+\phi'_{n-1,1}\delta m_1+\phi'_{n-1,2}\delta m_2+\&c.)$$
$$+A_2(\phi'_{n-2,0}\delta m_0+\phi'_{n-2,1}\delta m_1+\phi'_{n-2,2}\delta m_2+\&c.)$$
$$-A_3(\phi'_{n-3,0}\delta m_0+\phi'_{n-3,1}\delta m_1+\phi'_{n-3,2}\delta m_2+\&c.)$$
$$\&c. \qquad\qquad \&c.$$
$$=-(\lambda_0\delta m_0+\lambda_1\delta m_1+\lambda_2\delta m_2\ldots\ldots+\lambda_{n-1}\delta m_{n-1});$$
$$\therefore (A_0-A_1\lambda_{n-1}+A_2\lambda_{n-2}\ldots\ldots\pm A_n\lambda_0)\phi_n$$
$$=\lambda_0\delta m_0+\lambda_1\delta m_1+\lambda_2\delta m_2\ldots\ldots+\lambda_{n-1}\delta m_{n-1}+\delta m_n.$$

Hence, as the coefficient of δm_n in the value of ϕ_n is $\phi_{n,n}=\phi_{0,0}$, we must have

$$\phi_n=\phi_{0,0}(\lambda_0\delta m_0+\lambda_1\delta m_1+\lambda_2\delta m_2\ldots\ldots+\lambda_{n-1}\delta m_{n-1}+\delta m_n)\quad . \; . \;(12)$$

where the several coefficients are $\lambda_0\phi_{0,0}$, $\lambda_1\phi_{0,0}$, $\lambda_2\phi_{0,0}$, ... $\phi_{0,0}$, and these must be the same as the coefficients of ϕ_0, taken in a reverse order. But ϕ_0 is the same as the first supplementary difference in the scheme, for which a general formula has before been found.

If p denote the order of the differencing, counted from ϕ to the differences to be adjusted, then, according to (k), n being now $n+p$,

$$\phi_0=\Sigma\frac{x+1\ldots x+p-1}{n+p+1\ldots n+2p}\cdot\frac{n+1-x\ldots n+p-x}{2\ldots p-1}\delta m_x \quad . \; . \;(13)$$

As the coefficients of ϕ_n are those of ϕ_0 reversed, put $n-x$ for x, and

$$\phi_n=\Sigma\frac{n-x+1\ldots n-x+p-1}{n+p+1\ldots n+2p}\cdot\frac{x+1\ldots x+p}{2\ldots p-1}\delta m_x\ldots(14);$$

the respective coefficients of which are

$$\phi_{0,n}\;=\lambda_0\phi_{0,0}=p\frac{n+1\ldots n+p-1}{n+p+1\ldots n+2p}$$

$$\phi_{0,n-1}=\lambda_1\phi_{0,0}=p\frac{p+1}{1}\cdot\frac{n\ldots n+p-2}{n+p+1\ldots n+2p}$$

$$\phi_{0,n-2}=\lambda_2\phi_{0,0}=p\frac{p+1}{1}\cdot\frac{p+2}{2}\cdot\frac{n-1\ldots n+p-3}{n+p+1\ldots n+2p}$$

$$\phi_{0,n-3}=\lambda_3\phi_{0,0}=p\frac{p+1}{1}\cdot\frac{p+2}{2}\cdot\frac{p+3}{3}\cdot\frac{n-2\ldots n+p-4}{n+p+1\ldots n+2p}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\phi_{0,3}\;=\lambda_{n-3}\phi_{0,0}=\frac{p}{1}\cdot\frac{p+1}{2}\cdot\frac{p+2}{3}\cdot\frac{n-2\ldots n+p-3}{n+p+1\ldots n+2p}$$

$$\phi_{0,2}=\lambda_{n-2}\phi_{0,0}=\frac{p}{1}\cdot\frac{p+1}{2}\cdot\frac{n-1\ldots n+p-2}{n+p+1\ldots n+2p}$$

$$\phi_{0,1}=\lambda_{n-1}\phi_{0,0}=\frac{p}{1}\cdot\frac{n\ldots n+p-1}{n+p+1\ldots n+2p}$$

$$\phi_{0,0}=\frac{n+1\ldots n+p}{n+p+1\ldots n+2p}=\frac{(n+p)^2}{n(n+2p)}\phi'_{0,0}\ldots(15)$$

$$\therefore\lambda_0=\frac{p}{n+p}$$

$$\lambda_1=p\frac{p+1}{1}\cdot\frac{n}{(n+p)(n+p-1)}$$

$$\lambda_2=p\frac{p+1}{1}\cdot\frac{p+2}{2}\cdot\frac{n(n-1)}{(n+p)(n+p-1)(n+p-2)}$$

$$\lambda_3=p\frac{p+1}{1}\cdot\frac{p+2}{2}\cdot\frac{p+3}{3}\cdot\frac{n(n-1)(n-2)}{(n+p)(n+p-1)(n+p-2)(n+p-3)}$$

$$\vdots\qquad\qquad\vdots$$

$$\lambda_{n-3}=\frac{p}{1}\cdot\frac{p+1}{2}\cdot\frac{p+2}{3}\cdot\frac{n(n-1)(n-2)}{(n+p)(n+p-1)(n+p-2)}$$

$$\lambda_{n-2}=\frac{p}{1}\cdot\frac{p+1}{2}\cdot\frac{n(n-1)}{(n+p)(n+p-1)}$$

$$\lambda_{n-1}=\frac{p}{1}\cdot\frac{n}{n+p}$$

$$\lambda_n=1\quad\ldots\ldots\ldots\ldots\quad(16)$$

These values may be conveniently made to depend successively upon each other, as follows:—

$$\lambda_n=1 \qquad\qquad \lambda_0=\frac{p}{n+p}\lambda_n$$

$$\lambda_{n-1}=\frac{n}{1}\lambda_0 \qquad\qquad \lambda_1=\frac{p+1}{n+p-1}\lambda_{n-1}$$

$$\lambda_{n-2}=\frac{n-1}{2}\lambda_1 \qquad\qquad \lambda_2=\frac{p+2}{n+p-2}\lambda_{n-2}$$

$$\lambda_{n-3}=\frac{n-2}{3}\lambda_2 \qquad\qquad \lambda_3=\frac{p+3}{n+p-3}\lambda_{n-3}$$

$$\&c. \qquad\qquad \&c. \quad\ldots\ldots(17)$$

The coefficients are hence all easily deduced by means of the formulæ (12) and (10).

Moreover, according to the formula (10) the absolute value of the function ϕ_x is obviously $\Sigma(\lambda_x\phi_n)$, in which the terms under Σ are determined by giving to n the descending series of values n, $n-1$, $n-2$, &c. Hence from the foregoing general expressions for ϕ_n and the factors λ, we are thus enabled to obtain, for the

independent and direct calculation of any focal coefficient, the following general formula:—

$$\phi_{x,y}=\frac{n-x+1\ldots n-x+p}{2\ldots p-1}\cdot\frac{n-y+1\ldots n-y+p}{2\ldots p-1}\times$$

$$\Sigma\frac{(x+1\ldots x+p-1)(y+1\ldots y+p-1)}{n+1\ldots n+2p}\ldots(18)$$

where the terms under Σ are to be found by successively diminishing by unity every factor in both numerator and denominator. The number of these terms after the first will evidently be the lesser of the two values x, y. Also if $x'=n-x$, $y'=n-y$, be complementary numbers of the table, since then $\phi_{x,y}=\phi_{x',y'}$, the calculation is simplified, and consists of less terms, if we choose from these two equivalent forms that which contains the least of the four values x, y, x', y'.

The formula (18) expresses the value of the focal coefficient which belongs to a table constructed for $n+1$ differences. Let this, for the present, be distinguished by placing the symbol n as an ordinal exponent, and not as a power; thus $\phi^n_{x,y}$ designates the $y+1$th coefficient of the $x+1$th line of a table for $n+1$ values. Then, by applying the formula to express the several values of the following coefficients, viz.,

$$\phi^n_{x,y},\ \phi^{n-1}_{x-1,y-1},\ \phi^{n-2}_{x-2,y-2},\ \ldots\ \phi^{n-y}_{x-y,0},\ 0,$$

the leading factors, which precede Σ, will remain unaltered, but each succeeding series under Σ will become divested of its first term. The successive terms of the formula, taken separately, must, therefore, be the respective values of the first differences or decrements of the stated series of functions. Hence, by dividing the first of these terms by the second, we find that

$$\phi^n_{x,y}-\phi^{n-1}_{x-1,y-1}=\frac{n}{n+2p}\cdot\frac{x+p-1}{x}\cdot\frac{y+p-1}{y}(\phi^{n-1}_{x-1,y-1}-\phi^{n-2}_{x-2,y-2})$$

$$\therefore\ \phi^n_{x,y}=\phi^{n-1}_{x-1,y-1}+\frac{n}{n+2p}\cdot\frac{x+p-1}{x}\cdot\frac{y+p-1}{y}(\phi^{n-1}_{x-1,y-1}-\phi^{n-2}_{x-2,y-2})\ldots(19)$$

which supplies an easy method of calculating a coefficient for any table from those contained in the two tables immediately preceding it.

In extreme and exceptional cases, we have

$$\left.\begin{aligned}\phi^n_{x,1}&=\left(1+\frac{n}{n+2p}\cdot\frac{x+p-1}{x}\cdot\frac{p}{1}\right)\phi^{n-1}_{x-1,0}\\ \phi^n_{x,0}&=\frac{n+p}{n+2p}\cdot\frac{x+p-1}{x}\cdot\phi^{n-1}_{x-1,0}\\ \phi^n_{0,0}&=\frac{n+p}{n+2p}\cdot\frac{n+p}{n}\,\phi^{n-1}_{0,0}\end{aligned}\right\}\ldots(20)$$

For tables A, $p=4$; for tables B, $p=8$.

Examples.—In the table B for seven values, we have

$$\phi^6_{4,2}=\phi^5_{3,1}+\frac{6}{22}\cdot\frac{11}{4}\cdot\frac{9}{2}(\phi^5_{3,1}-\phi^4_{2,0})$$

$$=\frac{2778}{29393}+\frac{27}{8}\left(\frac{2778}{29393}-\frac{54}{4199}\right)$$

$$=\frac{10878}{29393}=\frac{1554}{4199}\left(=\frac{46620}{125970}\right);$$

$$\phi^6_{4,1}=\left(1+\frac{6}{22}\cdot\frac{11}{4}\cdot\frac{8}{1}\right)\phi^5_{3,0}$$

$$=7\frac{780}{29393}=\frac{780}{4199}\left(=\frac{23400}{125970}\right);$$

$$\phi^6_{4,0}=\frac{14}{22}\cdot\frac{11}{4}\cdot\phi^5_{3,0}$$

$$=\frac{7}{4}\cdot\frac{780}{29393}=\frac{195}{4199}\left(=\frac{5850}{125970}\right).$$

Also in relation to the tables B, the computed values of $\phi_{0,0}$, λ_0, λ_1, &c., for a few values of n are here appended:—

n	$\phi_{0,0}$	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
1	$\frac{9}{17.1430}$	$\frac{8}{9}$	1									
2	$\frac{5}{17.286}$	$\frac{4}{5}$	$\frac{8}{5}$	1								
3	$\frac{55}{17.19.78}$	$\frac{8}{11}$	$\frac{108}{55}$	$\frac{24}{11}$	1							
4	$\frac{33}{17.19.26}$	$\frac{2}{3}$	$\frac{24}{11}$	$\frac{36}{11}$	$\frac{8}{3}$	1						
5	$\frac{143}{17.19.70}$	$\frac{8}{13}$	$\frac{30}{13}$	$\frac{600}{143}$	$\frac{60}{13}$	$\frac{40}{13}$	1					
6	$\frac{91}{17.19.30}$	$\frac{4}{7}$	$\frac{216}{91}$	$\frac{450}{91}$	$\frac{600}{91}$	$\frac{540}{91}$	$\frac{24}{7}$	1				
7	$\frac{195}{17.19.46}$	$\frac{8}{15}$	$\frac{12}{5}$	$\frac{72}{13}$	$\frac{110}{13}$	$\frac{120}{13}$	$\frac{36}{5}$	$\frac{56}{15}$	1			
8	$\frac{130}{17.19.23}$	$\frac{1}{2}$	$\frac{12}{5}$	6	$\frac{132}{13}$	$\frac{165}{13}$	12	$\frac{42}{5}$	4	1		
9	$\frac{17.26}{19.23.45}$	$\frac{8}{17}$	$\frac{81}{34}$	$\frac{108}{17}$	$\frac{198}{17}$	$\frac{12.297}{13.17}$	$\frac{297}{17}$	$\frac{252}{17}$	$\frac{162}{17}$	$\frac{72}{7}$	1	
10	$\frac{17.18}{19.23.25}$	$\frac{4}{9}$	$\frac{40}{17}$	$\frac{225}{24}$	$\frac{220}{17}$	$\frac{330}{17}$	$\frac{396}{17}$	$\frac{385}{17}$	$\frac{300}{17}$	$\frac{180}{17}$	$\frac{40}{9}$	1

We shall conclude with a summary recapitulation of the general principles of calculation, according to the method propounded in the last investigation.

Let the given series of numbers, about the vicinity of any objectionable irregularities amongst the fourth differences, be differenced partially out to the twelfth order of differences. For example, in the following scheme, as in that on page 164, four refractory fourth differences (d_0, d_1, d_2, d_3) are considered to be centres of a disturbance which demands adjustment.

$$
\begin{array}{ccccccccccccc}
 & & & & d_{-4} & & & & & & & & \\
 & & & c_{-3} & & e_{-3} & & & & & & & \\
 & & b_{-2} & & d_{-3} & & f_{-2} & & & & & & \\
 & a_{-1} & & c_{-2} & & e_{-2} & & g_{-1} & & & & & \\
V_0 & & b_{-1} & & d_{-2} & & f_{-1} & & h_0 & & & & \\
 & a_0 & & c_{-1} & & e_{-1} & & g_0 & & i_0 & & & \\
V_1 & & b_0 & & d_{-1} & & f_0 & & h_1 & & k_0 & & \\
 & a_1 & & c_0 & & e_0 & & g_1 & & i_1 & & l_0 & \\
V_2 & & b_1 & & d_0 & & f_1 & & h_2 & & k_1 & & m_0 \\
 & a_2 & & c_1 & & e_1 & & g_2 & & i_2 & & l_1 & \\
V_3 & & b_2 & & d_1 & & f_2 & & h_3 & & k_2 & & m_1 \\
 & a_3 & & c_2 & & e_2 & & g_3 & & i_3 & & l_2 & \\
V_4 & & b_3 & & d_2 & & f_3 & & h_4 & & k_3 & & m_2 \\
 & a_4 & & c_3 & & e_3 & & g_4 & & i_4 & & l_3 & \\
V_5 & & b_4 & & d_3 & & f_4 & & h_5 & & k_4 & & m_3 \\
 & a_5 & & c_4 & & e_4 & & g_5 & & i_5 & & l_4 & \\
V_6 & & b_5 & & d_4 & & f_5 & & h_6 & & k_5 & & \\
 & a_6 & & c_5 & & e_5 & & g_6 & & i_6 & & & \\
V_7 & & b_6 & & d_5 & & f_6 & & h_7 & & & & \\
 & a_7 & & c_6 & & e_6 & & g_7 & & & & & \\
 & & b_7 & & d_6 & & f_7 & & & & & & \\
 & & & c_7 & & e_7 & & & & & & & \\
 & & & & d_7 & & & & & & & &
\end{array}
$$

Thus, by carrying out the differencing, the four twelfth differences m_0, m_1, m_2, m_3 are found; and after reversing their algebraic signs, according to equations (w), they will then become the values of δm_0, δm_1, δm_2, δm_3.

Now, by table B for four differences, the focal values are

$$\phi_0 = \frac{605\delta m_0 + 1320\delta m_1 + 1188\delta m_2 + 440\delta m_3}{277134}$$

$$\phi_1 = \frac{1320\delta m_0 + 3165\delta m_1 + 3048\delta m_2 + 1188\delta m_3}{277134}$$

$$\phi_2 = \frac{1188\delta m_0 + 3048\delta m_1 + 3165\delta m_2 + 1320\delta m_3}{277134}$$

$$\phi_3 = \frac{440\delta m_0 + 1188\delta m_1 + 1320\delta m_2 + 605\delta m_3}{277134}.$$

Having calculated these numerically, their expanded fourth differences are then to be developed as in the scheme on page 164, and

thus the sought values ($v_0, \ldots v_7$) of the corrections or adjustments of eight primitive numbers ($V_0, \ldots V_7$) are determined.

These adjustments will affect the values of twelve fourth differences ($d_{-4}, \ldots d_7$), which thereby become so modified and reduced in magnitude that the sum of their squares is less than would have similarly resulted from any other values of the focal quantities (ϕ_0, ϕ_1, ϕ_2, ϕ_3); and at the same time the values of the principal adjustments ($v_0, \ldots v_7$) are such that the sum of their squares is less than that of any other set of numbers that can produce, as fourth differences, the four middle adjustments (δd_0, δd_1, δd_2, δd_3).

If, according to the same method, second differences be used, instead of fourth, the operations will be perfectly analogous, but will, of course, involve less calculation. In the corresponding case of four centres, the schemes will be as follows:—

$$
\begin{array}{c|c|c|c|c|c|c}
 & & b_{-2} & & & & \\
 & a_{-1} & & c_{-1} & & & \\
V_0 & & b_{-1} & & d_0 & & \\
 & a_0 & & c_0 & & e_0 & \\
V_1 & & b_0 & & d_1 & & f_0 \\
 & a_1 & & c_1 & & e_1 & \\
V_2 & & b_1 & & d_2 & & f_1 \\
 & a_2 & & c_2 & & e_2 & \\
V_3 & & b_2 & & d_3 & & f_2 \\
 & a_3 & & c_3 & & e_3 & \\
V_4 & & b_3 & & d_4 & & f_3 \\
 & a_4 & & c_4 & & e_4 & \\
V_5 & & b_4 & & d_5 & & \\
 & a_5 & & c_5 & & & \\
 & & b_5 & & & &
\end{array}
\qquad
\begin{array}{l}
\delta f_0 = -f_0 \\
\delta f_1 = -f_1 \\
\delta f_2 = -f_2 \\
\delta f_3 = -f_3
\end{array}
$$

$$
\begin{array}{cc|c|c|c|c|c|c|c}
 & & & & \delta b_{-2} & & & & \\
 & & & a_{-1} & & \delta c_{-1} & & & \\
 & & v_0 & & \delta b_{-1} & & \delta d_0 & & \\
 & w_0 & & a_0 & & \delta c_0 & & \delta e_0 & \\
\phi_0 & & v_1 & & \delta b_0 & & \delta d_1 & & \delta f_0 \\
 & w_1 & & a_1 & & \delta c_1 & & \delta e_1 & \\
\phi_1 & & v_2 & & \delta b_1 & & \delta d_2 & & \delta f_1 \\
 & w_2 & & a_2 & & \delta c_2 & & \delta e_2 & \\
\phi_2 & & v_3 & & \delta b_2 & & \delta d_3 & & \delta f_2 \\
 & w_3 & & a_3 & & \delta c_3 & & \delta e_3 & \\
\phi_3 & & v_4 & & \delta b_3 & & \delta d_4 & & \delta f_3 \\
 & w_4 & & a_4 & & \delta c_4 & & \delta e_4 & \\
 & & v_5 & & \delta b_4 & & \delta d_5 & & \\
 & & & a_5 & & \delta c_5 & & & \\
 & & & & \delta b_5 & & & &
\end{array}
$$

and the formulæ for finding the focal values from the reversed sixth differences will be, according to the table A for four values, viz.:—

$$\phi_0 = \frac{49\delta f_0 + 84\delta f_1 + 70\delta f_2 + 28\delta f_3}{462}$$

$$= \frac{7\delta f_0 + 12\delta f_1 + 10\delta f_2 + 4\delta f_3}{66},$$

$$\phi_1 = \frac{84\delta f_0 + 177\delta f_1 + 164\delta f_2 + 70\delta f_3}{462},$$

$$\phi_2 = \frac{70\delta f_0 + 164\delta f_1 + 177\delta f_2 + 84\delta f_3}{462},$$

$$\phi_3 = \frac{28\delta f_0 + 70\delta f_1 + 84\delta f_2 + 49\delta f_3}{462}$$

$$= \frac{4\delta f_0 + 10\delta f_1 + 12\delta f_2 + 7\delta f_3}{66}.$$

Whether any real advantage is derivable from this particular method at all commensurate with the labour and complication of the process, or whether the method itself admits of material improvement by the substitution of a different hypothesis, must for the present be left to be determined hereafter.

The former method, or that of effecting, under the best possible conditions, any prescribed small adjustments of fourth differences, is much more simple in its application, the focal values being found at once by using the coefficients contained in the first set of tables.

We may also here state, that the two practical rules given at the commencement, for the adjustment of absolute or prominent errors, are at all times useful for the verification and revision of consecutive calculations, and are indispensable for the preliminary adjustment of a series of numbers previous to interpolation, or as a preparatory for further adjustment by one of the methods just alluded to, if this be considered necessary.

Should the resulting adjustments happen to be of any considerable magnitude, the fact will indicate either that the assumed interval of the table is too great, or that the presumed approximation of the given series of numbers to the true values is not entitled to be much relied upon.

In verifying results of computation, it should be borne in mind that, though practically correct, certain slight deviations from the exact values are continually incurred by the necessity of cutting off the figures at a given place of decimals. The last digit being always put down at the nearest figure, the amount of inaccuracy from this cause can never exceed half an unit of the last digit, and, amongst the several values, it may be either in excess or defect. Hence, if three consecutive differences, according to the rule, indicate an error of only an unit in the last figure, the case may in general be passed over without suspicion of inadvertence, since a displacement to that extent may naturally be accounted for by the occasional occurrence of something like half an unit alternately in excess and defect of two consecutive numbers; but if the discrepancy

of the differences should indicate a primary displacement of more than one unit, we may then be assured that an error really exists. The practical observance of this rule will obviate any unnecessary and useless re-examination of the original calculations.

It may be right also to observe generally, that the investigations that have been given in this paper have an implied reference to the elimination of casual errors or incidental imperfections, and do not necessarily apply to errors, assignable to certain specific causes, which may admit of being expressed by some mathematical function; for in such case the variations involved are no longer promiscuous in their character, and they do not contribute to produce any marked irregularities in the progression of the tabular differences. Such, for instance, are continued errors of principle in calculation, and, in some cases, personal errors of observation, which undoubtedly should be obviated or detected independently by other means, without having recourse to any theory of differences.

NOTE.—After finding the supplementary fourth difference adjustments by the formulæ (k), an analytical expression for the corrected functions may be obtained by completing the scheme of differences of the original functions (V_0, V_1, . . . V_n), that is, by treating the previous functions as zeros, and thus prefixing the values of the corresponding supplementary differences (d_{-4}, d_{-3}, d_{-2}, d_{-1}), which are simply deduced by numerical subtraction. For then, by the formulæ (ϵ), the value of an adjusted function will be

$$V_m+v_m=\Sigma'\frac{(m-\overline{x+3})(m-\overline{x+2})(m-\overline{x+1})}{2.3}(d_x+\delta d_x),$$

$$\text{or } (V+v)_m=\Sigma'\frac{(m-\overline{x+3})(m-\overline{x+2})(m-\overline{x+1})}{2.3}(d+\delta d)_x,$$

in which the ordinal characteristic (x) begins with the antecedent value -4.

Consider the case in which the fourth differences are absolutely neutralized, or in which $\delta d_0=-d_0$, $\delta d_1=-d_1$, &c. Then, in developing the terms, under Σ', of this expression, all of them, after the first four, will vanish; because the adjusted fourth differences $d_x+\delta d_x$ become severally zero when x reaches the values 0, 1, 2, &c. Therefore, in this case,

$$\begin{aligned}(V+v)_m=&\frac{(m+1)(m+2)(m+3)}{2.3}(d+\delta d)_{-4}\\&+\frac{m(m+1)(m+2)}{2.3}(d+\delta d)_{-3}\\&+\frac{(m-1)m(m+1)}{2.3}(d+\delta d)_{-2}\\&+\frac{(m-2)(m-1)m}{2.3}(d+\delta d)_{-1}\ .\ .\ .\ .\ (x);\end{aligned}$$

and this algebraic function of the third degree in m is that which represents

the original series of values (V_0, V_1, . . . V_n) more approximately than any other algebraic function of the same degree.

And the most approximate algebraic function of any degree may be obtained in like manner, by employing differences of an order relatively one degree higher. The analogous geometrical problem would be the determination of that curve line of a given order which shall follow most closely upon the devious track indicated by a number of given points.

CORRESPONDENCE.

THINGS WORTH NOTING.

To the Editor of the Assurance Magazine.

SIR,—I regret to have observed of late a great falling off in that which I consider by no means the least important or least interesting department of the *Magazine*—I mean the Correspondence department. I feel sure that if those of us who are accustomed to derive information and instruction from your pages were duly to discharge the duty we owe to the *Magazine* in return, the department to which I refer would speedily show signs of renewed vitality. We all, in the course of our reading, meet from time to time with "Things worth noting," which although often apparently trivial in themselves, are yet frequently by no means destitute of interest, either from their suggestiveness, their throwing light on some point in the history of our science, their illustrating the peculiarities and comparative advantages of different methods of investigation, or the like. Now, why should we not offer, now and then, for editorial approval, if we have nothing better, a little "Budget" of such "Things" of the above or cognate character as have come under our notice? It is very conceivable that wholesome discussion might thereby not unfrequently be excited, and no small amount of instruction elicited.

The advantage to be derived by the adoption of the course I have ventured to suggest, would not, as all experience proves, be entirely, or even principally, on the side of the readers of the *Magazine*. The writers would largely participate. This is a trite point. Suffice it to say, that the information we acquire as to the amount and the completeness of our acquaintance with a subject when we try to write upon it, and the stimulus given to our endeavours to supply the deficiencies which then almost always come to light, furnish far more than a compensation for any amount of labour that the effort to put our ideas upon paper may have cost us.

"Example," we are often told, "is better than precept." I therefore propose here to exemplify, by a first instalment, what I mean by "Things worth noting."

1. The history of the celebrated formula, $a_x = vp_x(1 + a_{x+1})$, which assigns the value of an annuity upon (x) in terms of that of the corresponding benefit upon ($x+1$) is pretty well known, up to a certain point. Mr. Milne ("Introduction," pp. xv., xvi.) attributes its discovery to Thomas Simpson, who published it in 1742, and he further informs us that it was rediscovered by Euler, whose publication of it dates in 1760. Subsequent writers, as

Mr. Galloway (*Treatise on Probability*, p. 93), attribute the formula entirely to Euler. The late Mr. Farren, however, in his work on the *Rise and Early Progress of the Doctrine of Life Contingencies in England* (1844), satisfactorily shows that the formula really originated with De Moivre, in the first edition of whose *Treatise on Annuities*, published in 1725, it was given to the world. It was suppressed, however, in the subsequent editions; the author probably considering that it was unnecessary, as his celebrated "Hypothesis" enabled him to assign the value of any annuity independently.

So far, as I have said, the history of the formula is pretty well known. But I do not think it is so well known that it has also been attributed to the now famous Mr. George Barrett. In the Useful Knowledge Society's *Treatise on Probability* (written by Sir J. W. Lubbock and the late Mr. Drinkwater Bethune), on p. 36, after deducing the formula

$$a_{x+1}=\frac{a_x}{vp_x}-1,$$

the authors say:—"By means of this expression, *which appears first to have been noticed by Mr. Barrett*, the value of any annuity may be deduced from that which precedes or follows it." And on p. 37, after a misdescription of Dr. Price's method of computing a table of the values of annuities, they further say:—"This labour, though diminished by means of *the equation noticed by Mr. Barrett*, is still unnecessary," &c.

The foregoing statements rest on entire misconception. The formula here given is De Moivre's, but inverted; that is, the equation is solved for a_{x+1}, instead of a_x, in which state it is useless. The value of the formula as given by De Moivre consists in this, that knowing, as we always do, the value of an annuity on the oldest tabular age, we are enabled thence to deduce in succession the values of annuities on all the younger ages. To acquire the like power in connexion with the inverted formula, we should require to know the value of an annuity on the youngest age, which we have no means of doing but by going through a laborious process, which De Moivre's formula was expressly devised to supersede. And it happens, oddly enough, that the relation noticed by Mr. Barrett—for he did notice *a* relation, as we all know—is one that enables us, if we please, to dispense altogether with the formula above attributed to him.

2. A circumstance that was pointed out to me some years ago by Mr. Welton, I consider quite deserving to be put on record here. It is, that Milne's Problems XVIII. and XXVII., pp. 204 and 222, although differently enunciated and symbolized, are in reality the same. A little consideration will show that they are so; and if confirmation is needed, it will be found in the identity of the forms given for their solution. Mr. Milne does not seem to have been aware of this, for there is no reference from the one to the other, and the paragraphs cited in the demonstrations are different in the two cases. Indeed, I believe Mr. Welton, by whom alone the identity of the two problems seems to have been observed, told me that on calling Mr. Milne's attention to the matter, that gentleman expressed surprise that it should be so. The two problems, or rather the two forms of the problem, seem to have been arrived at by following different routes; and Mr. Milne, it would appear, omitted to notice that the two routes conducted to the same point.

I purpose to send you hereafter some more "Things worth noting."

But, I confess, I shall be very much disappointed if I am suffered to monopolize this department of the *Magazine.*

I am Sir,
Your most obedient servant,
P. GRAY.

Camden Town,
21st February, 1865.

ON THE TABLES OF DEFERRED ANNUITIES PUBLISHED BY THE NATIONAL DEBT OFFICE.

To the Editor of the Assurance Magazine.

DEAR SIR,—In August, 1861, I drew the attention of your readers to the remarkable discrepancy which exists between the true premiums, as deduced from the Government Tables at 3 per cent, and those charged by the Government on the purchase of those deferred annuities in which the premiums are "returnable," either on death or at the option of the purchaser at any time prior to the commencement of the annuity, pursuant to 16 & 17 Vict., cap. 45.

I am now induced to revert to the subject, for two especial reasons; the first being, that these premiums are, as I am informed, computed at $3\frac{1}{4}$ per cent., and not at 3 per cent. as assumed in my last letter, whereby the difference is *greater* than I had then stated; the second, because I did not then give the very simple method by which those premiums may be deduced from the materials furnished by the tables themselves—nor, in fact, as far as I am aware, has any method of deducing premiums returnable at the option, as well as on the death, of a purchaser, been hitherto published in any work on life annuities.

The problem then is, to find the single premium for an annuity during the remainder of a life (x) after n years, with the condition that the premium is "returnable," without interest, on death or at the option of the purchaser at any time prior to the commencement of the annuity.

As the premium (P_x) is repayable at any time during the term (n), but without interest, it must be considered from two points of view; firstly, as a sum held on trust to be ready whenever called for; and, secondly, as a fund yielding an annual income which (not being repayable under any circumstances) is to be applied year by year, during the term, in the purchase of an annuity deferred for n years; but at the expiration of the term of n years, the condition as to the return of P_x having ceased, it must itself be applied to the purchase of an immediate annuity on the life at its then increased age of $(x+n)$ years.

Let P_x=single premium "returnable" for an annuity of £1;
i =interest on £1 for a year;
$p_{x_{\overline{n|}}}$=the annual premium payable at the end of the year for assuring to x a deferred annuity of £1 after n years;
a_{x+n}=annuity on a life aged $(x+n)$;

then $P_x\cdot\dfrac{i}{p_{x_{\overline{n|}}}}$ =the amount of deferred annuity which can be assured by the conversion of the annual interest into an annual premium,

and $P_x\cdot\dfrac{1}{a_{x+n}}$ =the amount of annuity which can be obtained by sinking P_x at the end of the term.

By addition,

$$P_x\left(\frac{i}{p_{x_{\overline{n}|}}}+\frac{1}{a_{x+n}}\right)=1,$$

$$\therefore P_x=\frac{1}{\left(\frac{i}{p_{x_{\overline{n}|}}}+\frac{1}{a_{x+n}}\right)}.$$

By an easy transformation to the columnar notation,

$$P_x=\frac{N_{x+n}}{(N_x-N_{x+n})i+D_{x+n}}.$$

As in the tables which we are now considering, $p_{x_{\overline{n}|}}$ $\left(\text{being}=\frac{a_{x\overline{|}n}}{a_x-a_{x\overline{|}n}}\right)$ and a_{x+n} are given, the following table was obtained with great facility, from which it will be seen that in the extreme case of an annuity to 31 after 50 years, the premium charged by the Government is more than *three and a half times* the correct amount.

Deferred Annuities of £30, *Males, Single Premiums, returnable without Interest at any time prior to commencement of Annuity, pursuant to* 16 *&* 17 *Vict., cap.* 45.

Age at Entry.	Term.	Government Premiums.	True Premiums, computed on Data of Tables at 3½ per Cent.	Difference between Government Premiums and true Premiums.	Error per Cent.
		£	£	£	
21	After 10 years	403·250	394·650	8·600	2·179
"	" 20 "	261·750	242·670	19·080	7·863
"	" 30 "	156·875	133·611	23·264	17·411
"	" 40 "	89·625	62·622	27·003	43·120
"	" 50 "	44·375	21·864	22·511	102·959
31	After 10 years	360·250	352·800	7·450	2·112
"	" 20 "	216·	199·467	16·533	8·289
"	" 30 "	121·375	96·429	24·946	25·869
"	" 40 "	61·125	34·881	26·244	75·238
"	" 50 "	25·250	6·923	18·327	264·726
41	After 10 years	297·375	290·646	6·729	2·315
"	" 20 "	167·125	146·634	20·491	13·974
"	" 30 "	84·125	56·013	28·112	50·188
"	" 40 "	34·750	11·554	23·196	200·761
51	After 10 years	230·	220·467	9·533	4·325
"	" 20 "	115·875	91·128	24·747	27·156
"	" 30 "	47·875	20·550	27·325	132·968

The results given in the foregoing examples are so startling, that I have been induced to prepare a second table, showing what proportion of the whole annuity which is to be entered upon at the end of the term is assured by the yearly application of the interest on the single premium, and what proportion is yielded by sinking the single premium at the end of the term.

Age at Entry.	Term.	Annual Income from Interest on Single Premium.	Deferred Annuity which Annual Interest will assure.	Annuity which Single Premium will purchase at the end of the Term.	Sum of last two Columns, being Total Benefit secured.
(x).	n.	$P_x . i$.	$P_x . \frac{i}{p_{x\overline{n}\rvert}}$.	$P_x . \frac{1}{a_{x+n}}$.	
21	10 years	12·826	8·676	21·324	£30
„	20 „	7·886	15·327	14·673	„
„	30 „	4·342	20·209	9·791	„
„	40 „	2·035	24·073	5·927	„
„	50 „	·710	27·012	2·988	„
31	10 years	11·466	8·666	21·334	£30
„	20 „	6·482	15·385	14·618	„
„	30 „	3·134	20·867	9·133	„
„	40 „	1·133	25·233	4·767	„
„	50 „	·225	28·337	1·663	„
41	10 years	9·446	8·701	21·299	£30
„	20 „	4·765	16·080	13·920	„
„	30 „	1·820	22·344	7·656	„
„	40 „	·375	27·224	2·776	„
51	10 years	7·165	9·119	20·881	£30
„	20 „	2·961	17·545	12·455	„
„	30 „	·667	25·063	4·937	„

If not trespassing too much on your valuable space, I should like to give one practical example in illustration of the previous observations.

Let us then suppose the case of a man aged 25, possessed of a capital of £47. 18*s*. 4*d*., which yields him an annual income (at $3\frac{1}{4}$ per cent.) of £1. 11*s*. 2*d*.; either, or both, of which he is desirous of devoting to a provision for old age.

On turning to Table No. 2 (money returnable), he will find that his capital is exactly sufficient to provide an annuity of £20 after 41 years, with the option of having his money returned at any time prior to the commencement of the annuity.

But on turning to Table No. 3 (money not returnable), he will see that, by devoting the yearly income of £1. 11*s*. 2*d*, he can assure an annuity of £24. 18*s*. 8*d*., instead of £20, and not part with his capital at all; and he will further discover, on looking to the table of immediate annuities, that if he wish to part with his money at the end of the term he can purchase a further annuity of £5. 7*s*. 9*d*., making together £30. 6*s*. 5*d*., instead of £20—a result which will be found to be fully confirmed by my previous deductions.

I have now practically shown that a person can secure a deferred annuity under Table No. 3, greater than he can assure under Table No. 2 by 51·6 per cent., on precisely the same security and at precisely the same expenditure in both instances, with the additional advantage that in the former case he can at the end of the term exercise an option against the Government, should his then state of health render such a course desirable; but even this does not fully describe all the disadvantages which he suffers under Table No. 2, as circumstances, other than his state of health, may render it necessary for him to divert his capital into some other channel at the end of the term; should this be so, he will under Table No. 2 have secured

nothing in the shape of an annuity, whilst under Table No. 3 he will receive £24. 18*s*. 8*d*. per annum for the remainder of his life. In this illustration I have disregarded the provision of the Act, that no person can assure an annuity of more than £30, and also any rule which may exist as to insuring fractions of a pound, as these can have no bearing on the question before us; but it may, perhaps, be as well to point out that, as in the Government tables, the premiums for deferred annuities are, of course, payable at the beginning, whilst in the formula which I have given they are payable at the end of the year, the assurance under Table No. 3 must be made at the age of $x+1$ for $n-1$ years.

In conclusion, allow me to draw your attention to the table for deferred annuities (money not returnable), which has this week been laid before Parliament. In it are given tables of immediate annuities and tables of deferred annuities; with these materials, the values of annuities for terms of years are immediately obtained, with the following results:—

Age.	Males—Value of an Annuity of £1 for the Term of							
	10 Years.	12 Years.	14 Years.	16 Years.	18 Years.	20 Years.	22 Years.	24 Years.
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20 to 21	7 18 9	9 2 9	10 4 8	11 4 8	12 3 0	12 19 9	13 15 0	14 8 11
30 „ 31	7 19 4	9 3 5	10 5 5	11 5 5	12 3 9	13 0 5	13 15 6	14 9 2

in all of which cases it will be seen that the value of a life at 31 is *greater* than that of a life at 21; and similarly it may be shown that the value of an annuity for 10 years is precisely the same at the ages of 21 and 39.

I hope, on a future occasion, to be allowed to prove that the annual premiums payable under the "money returnable" scale are open to the same objections, although not to the same degree, as those which I have urged against the single premiums under a like condition.

I am, dear Sir,
Yours truly,

London, 25th February, 1865. J. W. STEPHENSON.

MR. SPRAGUE'S LETTER IN THE LAST NUMBER OF THE JOURNAL.

To the Editor of the Assurance Magazine.

Sir,—It was my intention to have asked you to be good enough to insert a reply, which I have written at some length, to the letter of Mr. Sprague which appeared in the last Number of the *Magazine*. But some of my friends, on whose judgment I place great reliance, who have seen the manuscript, urge me to withdraw it, on the ground that the affairs of particular Societies ought not, under any circumstances, to be discussed in the *Journal of the Institute*. For this reason, and for this reason only, I abstain from entering into further controversy on the subject.

I have the honour to be,
Sir,
Your obedient servant,

10*th March*, 1865. ARTHUR H. BAILEY.

MR. MAKEHAM'S LETTER IN THE LAST NUMBER OF THE JOURNAL.

To the Editor of the Assurance Magazine.

DEAR SIR,—Your last Number contains a letter from Mr. Makeham on the subject of some remarks made by me at a meeting of the Institute of Actuaries, when his paper "On the solution of general problems in survivorships" was read.

It is difficult to understand Mr. Makeham's object in writing the letter, or what interest the discussion he seems desirous of raising can have for your readers. Certainly, if it were not out of respect for them I should not trouble you with any reply upon the subject.

My statement was, that having had frequent occasion to apply Milne's formulæ for the solution of cases of survivorship among three lives, I had found the common method of approximating, by taking an equivalent single life for two out of the three lives, to give results that came surprisingly near to those obtained by the complete methods.

The remark, which was merely an incidental one, was not urged either in opposition to or in disparagement of anything Mr. Makeham had advanced; and if he doubt the truth of it, he can easily satisfy himself upon the subject by making the requisite calculations.

I am, dear Sir,
Very faithfully yours,
W. B. HODGE.

Campden Hill, W.
6th March, 1865.

SUGGESTION AS TO THE COMMUNICATION TO THE JOURNAL OF CASES OF UNUSUAL CHARACTER OR OF DIFFICULT SOLUTION.

To the Editor of the Assurance Magazine.

SIR,—I have often thought that much additional interest would be conferred upon the *Assurance Magazine*, if some of your more experienced supporters would occasionally communicate their opinions upon such cases submitted to them, as may be of unusual character or of more than ordinary difficulty. There can be no doubt that such a course would be of very great advantage to the younger members of our profession; and I think it not improbable that it might have the effect of reducing to some extent those wide divergencies of opinion which at present occasionally occur among actuaries.

With these objects in view I subjoin the particulars of a case recently submitted to myself; and would hope that it may call forth solutions from several of your readers in your next Number; in which case I will state the solution I arrived at myself.

Case.—A freehold estate is settled for life upon persons of the ages 55, 53, 51, 50, 48, in succession; and the survivor of the five is entitled to the reversion to the fee simple. The present tenant for life (55) wishes to buy out the reversioners; and, with that view, inquires what he may properly pay for the interest of the youngest life, 48?

Your obedient servant,
JUVENIS.

February, 1865.

NEW GERMAN PUBLICATIONS.

To the Editor of the Assurance Magazine.

SIR,—Allow me to direct your attention to two valuable recent publications:—

1. *On the Law prevailing in the Acts apparently under the control of the Human Will, from a Statistical point of view* (*Die Gesetzmässigkeit in den scheinbar willkührlichen menschlichen Handlungen vom Standpunkte der Statistik*). By Adolph Wagner; in two Parts: Hamburg, 1864; Boyes & Geisler.

The first part takes a more general view of the subject. The author examines whether there is a law existing in the acts of man. He shows, in accordance with all authorities who ever attempted the subject, that the existence of such laws cannot be doubted. He tries to fix the character of these laws, and to establish the methods which lead to their discovery. He shows the truth of his assertions in the statistics about marriages, suicides, and crimes. The second part deals with the facts more in detail, it contains a great number of statistical data, especially referring to suicides, which the greater part of the book is devoted to. You will find many very interesting tables elucidating the subject, and putting it before you in a new light. Some of the principal results which the author derives from his observations are—the numbers of suicides in the different countries of Europe are varying very little from year to year; suicide is increasing at present in Europe more rapidly than the population; the seasons are of great influence on the frequency of suicide, so is the sex, but in this the various countries differ; age is of uniform influence in all Europe, so is descent, nationality, and religion, while the promotion of knowledge exercises no visible influence.

2. *On Assurance against permanent Incapability; a complete set of Tables for the Computation of Premiums, and the Sums to be reserved for Assurance against Invalidity.* By Dr. Aug. Wiegand, Director of the Life Assurance Company Iduna at Halle: 1865, Herm. Berner.

The theory of invalidity, or permanent incapability, has been established a few years ago by Dr. Heym, at Leipsic, and lately it is more and more acknowledged from all parts that the former institutions to provide against it, which have been established without any sound foundation of calculation, give not the least security to their members. At the same time, associations of all kinds are formed to realize the principle of social self-relief, and, in consequence, a wish for institutions to provide against invalidity is generally felt. The publication of Dr. Wiegand's treatise will therefore be of great benefit to all those who take an interest in the promotion of these ideas. Dr. Wiegand himself is one of the principal directors of the Invalid Institution for Physicians at Halle, and well known for his able essays on life assurance and mathematics.

The present work contains all the necessary formulæ for the computations in this branch, and a complete set of tables at $3\frac{1}{2}$ and at 4 per cent. interest.

I can well recommend the two books to every one interested in the subject.

I am, Sir,

Yours most truly,

WILHELM LAZARUS.

Hamburg, February, 1865.

ON THE ADAPTATION OF ASSURANCE FORMULÆ TO THE ARITHMOMETER OF M. THOMAS.

To the Editor of the Assurance Magazine.

SIR,—In adapting assurance formulæ to the processes that may be wrought on M. Thomas de Colmar's Arithmometer, the following expressions have occurred to me. They appear to be worthy of record, and I therefore submit them to you.

For simplicity of notation, let

$l=l_x$ or $l_{x.y}$, &c., *i.e.*, expressions into which only *lives* enter;
$d=d_x$ or $d_{x.y}$, or $d_x l_{y+\frac{1}{2}}$, &c., expressions into which *deaths* enter;
$r=(1+r)l_x$ or $(1+r)l_{x.y}$, &c., expressions into which the *rate* of interest enters;
$a=a_x$ or $a_{x.y}$, &c., *annuities* of all kinds;
$A=A_x$ or $A_{x.y}$, or $A_{\overset{1}{\overline{x.y}}}$, &c., *assurances* of all kinds;

also let l_1, a_1, A_1, be the same, but *advanced one year;*

then in all cases $$a=\frac{a_1 l_1+l_1}{r} \quad \ldots \ldots \quad (1)$$

and $$A=\frac{A_1 l_1+d}{r} \quad \ldots \ldots \quad (2)$$

Of these equations, the first, in the form $a=\frac{(1+a_1)l_1}{r}$, is well known; the second is, to me at least, new; and both, for mechanical computation, are very convenient. Being symmetrical they are easy to remember.

The subjoined small table will enable any of your readers, who may be so disposed, to try these methods. I do not propose them as the fittest for working *by hand*, but I am persuaded that the days of hand work in the actuary's craft are coming to an end. The arithmometer is not an expensive machine, and its speed and certainty are invaluable.

I am, Sir,
Yours very truly,
J. HANNYNGTON.

Kingstown (near Dublin),
18*th March*, 1865.

Carlisle Table, 3 per Cent. Difference of Ages, 3 Years.

x.	d_x.	l_x.	$l_{x+\frac{1}{2}}$.	$d_x l_{y+\frac{1}{2}}$.	$l_{x.y}$	$(1+r)l_x$	$(1+r)l_{x.y}$	y.	a_x.	$A_{\overset{1}{\overline{x.y}}}$.
98	3	14	12·5	79·5	420	14·42	432·60	95	2·3883366	·4829954
99	2	11	10	41	253	11·33	260·59	96	2·1308922	·5116357
100	2	9	8	32	162	9·27	166·86	97	1·6825565	·5699206
101	2	7	6	25	98	7·21	100·94	98	1·2281855	·6438465
102	2	5	4	20	55	5·15	56·65	99	0·7710435	·7270884
103	2	3	2	16	27	3·09	27·81	100	0·3236246	·7847984
104	1	1	0·5	6	7	1·03	7·21	101	0·0000000	·8321775

$$a_x=\frac{(1+a_{x+1})l_{x+1}}{(1+r)l_x}; \quad A_x=\frac{A_{x+1}l_{x+1}+d_x}{(1+r)l_x}; \quad A_{\overset{1}{\overline{x.y}}}=\frac{A_{\overset{1}{\overline{(x.y)+1}}}\,l_{(x.y)+1}+d_x l_{y+\frac{1}{2}}}{(1+r)l_{x.y}}.$$

At the highest age l_{x+1} and a_{x+1} both vanish, and at the age next below a_{x+1} vanishes; the initial value is, therefore, $\frac{l_{x+1}}{(1+r)l_x}$, where $x=103$.

At the highest age $A_{\overset{1}{\overline{(x.y)+1}}}$ vanishes, and the expression becomes $\frac{d_x l_{y+\frac{1}{2}}}{(1+r)l_{x.y}}$.

THE

ASSURANCE MAGAZINE,

AND

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the Statistics of Second Marriages among the Families of the Peerage. By ARCHIBALD DAY, ESQ., *Actuary of the London and Provincial Law Assurance Society, and Fellow of the Institute of Actuaries.*

[Read before the Institute, 27th March, 1865.]

ON a former occasion I had the honour of submitting to the consideration of the members of the Institute of Actuaries some statistics respecting marriages amongst the families of the peerage, from which were deduced tables showing the probabilities of marriage, whether for the first or second time, according to the age of the husband. The subject would, perhaps, have been better fitted for a discussion at a meeting of the Statistical Society, had it not been brought within the range of assurance topics by the avowed object for which the statistics were collected, viz., to throw some additional light on a class of risks which were every day becoming more numerous and important—the class of assurances to protect contingent reversions which would pass from presumptive heirs in the event of issue being born to a present tenant for life. They are now more familiarly known as "issue risks."

It is believed that some progress towards the solution of questions of this character was made by that investigation. A step in advance may have been taken, but the inquiry could not be con-

sidered complete without an attempt to analyze, if possible, some of the results arising from the marriages contracted; I mean, more especially, the duration thereof, their fruitfulness, and the proportions of the sexes of the children. There has been no diminution of interest in the subject since the publication of my former paper, and the great increase in the number of assurances against the risks above referred to has called peremptorily for a further investigation.

Fully convinced that the peerage families and the aristocracy generally furnish the most valuable, the most trustworthy, and, at the same time, the most easily collected data, upon which to found observations of this character, I have devoted some considerable time to the labour of extracting from the books of the peerage the statistics of as many marriages as would supply sufficient materials for the present purpose. In the paper, prepared jointly by Mr. Bailey, the Actuary of the London Assurance, and myself, "On the Rate of Mortality prevailing amongst the Families of the Peerage," we gave some reasons for our belief in the general accuracy of the facts recorded in these books, and more especially in the publication known as *Lodge's Peerage*, and this has been the source from which I have principally drawn my information on the present occasion. From the *Baronetage* and the *Landed Gentry* I have obtained some further cases; but I did not feel that so much reliance could be placed on these generally, and I have therefore almost entirely confined my selection from them to those members of the baronetage and of the landed gentry who have married into peerage families. In addition to these sources, I have been presumptuous enough to include a Family Register, most carefully and, I had almost said religiously, kept by a personal relative, in which are most minutely recorded the exact dates of birth, marriage, and death, in nearly 100 families, the descendants of a gentleman formerly resident in the county of Kent. The longevity of this family is sufficiently remarkable to justify a brief record.

Mr. B. married at 25 years of age. He had 13 children, and died at the age of 75, leaving 12 surviving him. Mrs. B., his widow, died at the age of 94. Of the 12 surviving children, 4 have since died, at the respective ages of 88, 79, 79, and 58. The remaining 8 still survive, 3 of whom exceed 80 years of age, 4 are above 70, and the youngest is 68.

Mrs. B. had also three sisters who exceeded 80, dying respectively at the ages of 88, 83, and 81; and a brother who died at 79.

I have thus obtained records of 3,417 marriages in all, which are divided as follows:—

First marriage of husbands . .	2,857
Second ditto ditto . .	507
Third ditto ditto . .	50
Fourth ditto ditto . .	3
	3,417

To insure as much accuracy as possible, the facts, after having been primarily extracted from *Lodge's Peerage*, 1856—compared at the same time with Burke and Debrett — were checked by examination with the 1864 edition of Lodge, from which also additional materials were obtained. By the courtesy of the President of the Institute I had also the opportunity of seeing a manuscript volume of extracts of the same nature, made by Mr. Le Cappelain for the late Albion Assurance Company, which proved of some little assistance. I believe this investigation has never been completed.

My extracts were made upon a series of cards of somewhat similar form to those recently employed for obtaining the mortality experience of Assurance Companies. A facsimile is subjoined, of which but little explanation seems needed:—

2M. No. 82 (Leicester).

Husband's Age. 67. *Wife's Age.* 18.

Married . . . . 1822.3.
D. Died . . . . 1842.7.
Duration of Marriage . . 20/4.

CHILDREN BORN.

	BOYS.	IF DEAD.	PERIOD SINCE MARRIAGE.	GIRLS.	IF DEAD.
1	1823.1		/10		
2	1824.9		2/6		
3	1827.1		4/10		
4	1828.7		6/4		
5			10/	1832.3	
6	1835.2	D.	12/11		
7					
8					
9					
10					
11					
12					

The number and title are for reference only. 2M above the husband's age is to explain that the facts on the card refer to his second marriage. The age is the *age last birthday* at the time of marriage. The dates are in years and months, and the children born are placed in their order of birth—the boys on the left and the girls on the right of the card (still-born births have been everywhere excluded). The D against the date of death is placed on the left or right side, according as the marriage terminated by death of husband or wife. Space is allowed for recording the birth of twelve children; and in the few instances in which that number was exceeded they were continued on the back of the card.

These particulars could not be completely furnished for all the 3,417 families under observation—a large portion of the marriages being still existing on the 31st December, 1863 (the date at which the observations close). In many instances the age of the wife could not be ascertained; in some, the number of children was doubtful; in others, the dates of their births or the order in which they were born, were deficient; still, in every one there was some information that was available for one branch or other of the contemplated investigation; and in the future it is hoped that many of the incomplete records may be supplied.

Having obtained this large number of facts, it very soon became clear to me that the manipulation of them for all the purposes I had designed would involve more time and labour than for various and sufficient reasons I could immediately command, and it was therefore necessary to decide whether the paper I had in contemplation should be postponed to a future day, or whether I might offer to the Institute an instalment thereof, with the hope of completing the subject at no very distant period. Seeing that there must necessarily be a broad line of demarcation between the results of first and subsequent marriages, and that such a division was especially convenient when the motive for taking up the investigation at all was considered, I determined to complete the most important, although the shortest part of my task, and have therefore compiled a chapter on the *second* marriages of husbands in the families of the peerage, leaving to a future day what should properly perhaps have been the first division of the subject.

The number of second and subsequent marriages under observation (560) is rather large in proportion to the total number of marriages (3,417)—being equivalent to 16·4 per cent.—whereas the proportion in the population of England, according to the last Return of the Registrar-General, 1862, was

13·7 per cent.—or on average of ten years, 13·9 per cent. It was shown in the former paper that the marriage rate among widowers in the peerage families, from age 50 upwards, was greater than the corresponding rate in the general population, and this result might therefore have been anticipated. On the other hand, it is just possible that from a natural anxiety to obtain as large a number of second marriages as practicable, I may perhaps have included a few cases in which the statistics of the first marriage were deficient, and thereby have very slightly increased the proportion.

In the very valuable paper of Mr. Hendriks, on the vital statistics of Sweden,* will be found a vast collection of facts relating to the conjugal condition of the people of that kingdom, and from one of these tables I am able to quote the following proportions of first and subsequent marriages of husbands in Sweden between the years 1821 and 1855. It will be seen that there is a striking agreement in the proportions of marriages in the populations of the two countries.

Percentage of Marriages.

	Peerage.		Sweden.		England.
First marriages........	83·64		86·7802		86·1
Second do.	14·85	16·36	12·5393	13·2198	13·9
Third do.	1·43		·6326		
Fourth do.	·08		·0436		
Fifth do.	..		·0041		
Sixth do.	..		·0002		
	100·00		100·0000		100·0

The number of second marriages being necessarily small, I have, in all the tables illustrating this paper, grouped them in quinquennial years of age, which will be found sufficiently near for all practical purposes, and subsequent marriages have all been classed in the same category as second marriages.

One of the first elements in the consideration of the anticipated fertility of a marriage will naturally be the age of the bride, and I have accordingly prepared a table (A) exhibiting the ages of the wives at the date of marriage, where the records supply them. It appears that there were 11 cases only where the ages of the husbands were exceeded by those of their wives, and in all these the difference was comparatively trifling, not exceeding on the average three years.

Of the 560 remarriages of widowers, in Table A, it will be seen

* See *Journal of the Statistical Society*, vol. xxv.

that in 274 instances—nearly 50 per cent. of the whole—the ages of their wives had been ascertained; and of these, 10 only, or about 4 per cent., were of such an age (50 and upwards) that issue of the marriage might be deemed highly improbable; whence it follows, that in the calculation of a premium for assurance against issue, it is only safe to assume that in the event of a husband contracting a second marriage it will be with a lady of child-bearing age.

From the Return of the Registrar-General for the year 1862, I find 13,311 marriages of widowers, where the ages of both husband and wife are given, and the proportion of wives who exceeded 50 years of age at marriage is 10·73 per cent., which is higher than that deduced above from the experience of the peerage. The comparison is carried further in the lowest lines of the table, the percentage of marriages being given for every quinquennial group of age of the wife, whence it appears that out of 100 remarriages of widowers those who married wives under 20 are four times as many in the peerage as in the general population; between 20 and 25, nearly twice as many; and from 25 to 30, there is still a considerable difference in favour of the peerage; showing conclusively that the aristocracy, in their second marriages, obtain much younger partners than their contemporaries of the general population, arising, probably, from ladies marrying for fortune or position.

I am not sure that the remaining 286 wives in the table, whose ages were not known, could be distributed in the same proportions; indeed, it is more reasonable to assume that they would, on the average, be older, and this is borne out by other considerations which will appear hereafter.

It will be noticed that there are records of two octogenarian bridegrooms. They were aged respectively 80 and 83, at the date of their second marriage.

Of the three *fourth* marriages previously referred to, it is rather remarkable that the ages of the husbands were comparatively young. They were 47, 51, and 51 respectively. The youngest became a widower again at 62, and was contented to remain in that condition, although he survived his last wife some years.

The final columns in Table A exhibit respectively the average ages of the wives of husbands in each quinquennial group, and the average number of years which the husband's age exceeded that of the wife. The Report of the Census Commissioners, for 1861, quotes the difference of age between husband and wife as $2\frac{1}{2}$ years only; but by far the greatest number of these were undoubtedly *first* marriages.

TABLE A (*Peerage Families*).—*Remarriages of Widowers and Comparative Ages of Wives.*

Ages of Husbands.	Ages of Wives.										No. of Re-marriages where Wives' Ages are known.	No. of Re-marriages where Wives' Ages not recorded.	Total Number of Re-marriages.	Average Age of Wives.	Difference between Ages of Husband and Wife.	Ages of Husbands.
	15–	20–	25–	30–	35–	40–	45–	50–	55–	70–						
24–29	3	9	5	2	..	..	..	..	..	..	19	14	33	23·5	3·0	24–29
30–	7	17	5	2	..	..	..	..	..	..	31	27	58	22·3	9·7	30–
35–	5	24	14	4	5	2	..	..	..	..	54	34	88	25·7	11·3	35–
40–	1	14	14	9	5	2	2	2	..	..	49	44	93	29·7	12·3	40–
45–	..	7	12	3	5	2	1	..	..	..	30	44	74	29·6	17·4	45–
50–	2	1	7	2	5	3	4	..	..	..	24	32	56	33·6	18·4	50–
55–	..	4	5	8	4	4	2	..	2	..	29	34	63	34·6	22·4	55–
60–	..	3	..	4	1	3	5	1	1	..	18	23	41	38·9	23·1	60–
65–	1	1	1	..	2	3	2	1	..	..	11	18	29	37·5	29·5	65–
70–	..	1	1	1	..	2	1	1	1	..	8	12	20	40·1	31·9	70–
75–	..	..	..	..	..	..	..	..	..	1	1	2	3	..	..	75–
80–	..	..	..	..	..	..	..	..	..	..	..	2	2	..	..	80–
	19	81	64	35	27	21	17	5	4	1	274	286	560			
Percentage of marriages (Peerage)	6·93	29·56	23·35	12·77	9·86	7·66	6·21	1·83	1·46	·37	100·00					
Do. (England) 1863	1·76	15·38	17·99	16·40	15·15	13·50	9·09	5·52	2·86	60 & upwds. 2·35	100·00					

Rather as a matter of curiosity and of interest than of practical importance, I have prepared a table (B) to illustrate the proportions of widowers who married spinsters and widows respectively.

From this it would seem that although there is a great deal of irregularity in the marriages of the younger widowers, the general and perhaps natural conclusion is, that in proportion as the ages of the bridegrooms increase they select partners from ladies who have had former husbands.

Distinguishing between second and third marriages, I find that second marriages are in the following proportions:—

With spinsters	85·0 per cent.
With widows	15·0 "

whilst in third marriages the proportions are—

With spinsters	77·4 per cent.
With widows	22·6 "

In the table I have placed in juxtaposition with the proportions amongst peerage families, those of the widowers in England who married in 1862, where the ages were distinguished (13,311 out of 22,457). There is much greater regularity here, arising, no doubt, from the larger numbers observed. The year 1862 does not appear to have been exceptional in this respect, for the average of 18 years, according to the Registrar-General, was—

Widowers with spinsters	(18 years)	65·68	(1862)	63·7
Widowers with widows	"	34·32	"	36·3
		100·00		100·0

I cannot compare the proportions of the Swedish marriages according to ages, but the total number of second marriages, on an average of 45 years, appear to have been in the proportion of 76·6 with spinsters to 23·4 with widows. For the single year 1855 the proportions were slightly different, being with spinsters 81·4 per cent., and with widows 18·6 per cent.

Table C exhibits the *duration* of remarriages, and here second have been distinguished from subsequent marriages. The former, it will be seen, exceed the latter on the average by 3½ years' duration only. I desire to call attention to the column representing the average duration of the remarriages with wives of whose ages there was no record. For every quinquennial group but two the duration is less than in those where the wives are known to be younger than their husbands, and it confirms the assumption, made in a former part of the paper, that this class cannot be distributed

TABLE B.—*Remarriages of Widowers—Proportions married to Spinsters and Widows.*

Ages of Husbands.	Peerage Families.				England, 1862 (13,311 Marriages where Wives' Ages are recorded).		Sweden.		Ages of Husbands.
	Number who married Spinsters.	Number who married Widows.	Proportion per Cent. married Spinsters.	Proportion per Cent. married Widows.	Proportion per Cent. married Spinsters.	Proportion per Cent. married Widows.	Proportion per Cent. married Spinsters.	Proportion per Cent. married Widows.	
24–29	30	3	90·84	9·16	90·2	9·8	Average, 45 years .. 76·6	Average, 45 years .. 23·4	24–29
30–	56	2	96·56	3·44	80·5	19·5	1855 .... 81·4	1855 .... 18·6	30–
35–	81	7	92·04	7·96	71·2	28·8			35–
40–	79	14	84·94	15·06	61·3	38·7			40–
45–	67	7	90·55	9·45	52·2	47·8			45–
50–	49	7	87·50	12·50	46·1	53·9			50–
55–	49	13	79·03	20·97	39·6	60·4			55–
60–	29	13	69·05	30·95	36·2	63·8			60–
65–	19	10	65·52	34·48	29·8	70·2			65–
70–	12	8	60·00	40·00	38·2	61·8			70–
75 & upwards	1	4	20·00	80·00	34·1	65·9			75 & upwards
Total	472	88	84·28	15·72	63·7	36·3			

TABLE C (*Peerage Families*).—*Remarriages of Widowers* (*Duration thereof*).

Ages of Husbands.	Wives younger than Husbands. No. of Marriages.	Wives younger than Husbands. Duration.	Wives younger than Husbands. Average Duration.	Wives whose Ages are not recorded. No. of Marriages.	Wives whose Ages are not recorded. Duration.	Wives whose Ages are not recorded. Average Duration.	Wives older than Husbands. No. of Marriages.	Wives older than Husbands. Duration.	Wives older than Husbands. Average Duration.	Total. No. of Marriages.	Total. Duration.	Total. Average Duration.
Second Marriages.												
		Years. mos.	Years. mos.		Years. mos.	Years. mos.		Years. mos.	Years. mos.		Years. mos.	Years. mos.
24–29	13	355 9	27 4	10	251 11	25 2	4	72 10	18 2	27	680 6	25 2
30–	24	511 6	21 4	12	210 8	17 7	..	..	..	36	722 2	20 0
35–	35	774 4	22 1	27	385 11	13 4	..	..	..	62	1160 3	18 8
40–	27	471 10	17 6	29	532 5	18 4	2	41 1	20 7	58	1045 4	18 0
45–	14	260 10	18 7	25	459 1	18 4	..	..	..	39	719 11	18 5
50–	14	232 11	16 7	17	234 1	13 9	..	..	..	31	467 0	15 1
55–	17	236 11	13 11	19	204 11	10 9	..	..	..	36	441 10	12 3
60–	11	114 10	10 5	16	148 3	9 3	..	..	..	27	263 1	9 9
65–	10	125 2	12 6	11	108 2	9 10	..	..	..	21	233 4	11 1
70–	6	74 1	12 4	6	40 2	6 8	..	..	..	12	114 3	9 6
75 & upwds.	1	1 1	1 1	3	7 9	2 7	..	..	..	4	8 10	2 2
	172	3159 3	18 4	175	2583 4	14 9	6	113 11	19 0	353	5856 6	16 7
Third and Fourth Marriages.												
30–	2	33 7	16 9	2	59 8	29 10	..	..	..	4	93 3	23 4
40–	7	98 7	14 1	5	63 0	12 7	2	37 9	18 11	14	199 4	14 3
50–	3	30 2	10 1	6	107 9	17 11	..	..	..	9	137 11	15 4
60–	3	30 9	10 3	2	8 10	4 5	..	..	..	5	39 7	7 11
70–	1	5 1	5 1	4	12 8	3 2	..	..	..	5	17 9	3 6
	16	198 2	12 5	19	251 11	13 3	2	37 9	18 11	37	487 10	13 2

according to age in the same ratio as those whose ages have been ascertained.

Amalgamating the second marriages with those subsequent, and extracting from the table the remarriages above 50 years of age, which may be taken to represent the ages of those who are the subjects of the contingent assurances that this paper is intended to illustrate, it appears that there are records of 150 remarriages, the total duration of which was 1,723 years 7 months, or on the average 11½ years to each marriage—or assuming the age of 60, as more nearly correct for our purposes, the average duration of marriage is nine years and a fraction—an ample period for the birth of heirs.

On the other hand, some of the marriages seem to have been of very short duration. The observations on lives above 60 record three which terminated within a twelvemonth, all fatally for the husband—viz., at 3, 6, and 8 months respectively. Thirteen are under 3 years' duration, of these two only terminated by the death of the wife.

As might naturally have been anticipated, by far the larger proportion of these remarriages terminate by the death of the husband, the excess in age being, as was shown before, so much on his side. We may be prepared, therefore, for the results exhibited in Table D, where, speaking generally, the deaths of husbands to those of wives are in the proportion of two-thirds to one-third.

TABLE D (*Peerage Families*).—*Remarriages of Widowers, and Proportions terminated by Death of Husband or Wife.*

Ages of Husbands.	No. of Marriages.	Terminated by Death of Husband.	Terminated by Death of Wife.	Terminated by Divorce.	Per Cent. of Husbands Dead.	Per Cent. of Wives Dead.	Per Cent. of Divorces.
24–29	27	10	15	2	37·0	55·6	7·4
30–	102	60	40	2	58·8	39·2	2·0
40–	111	68	42	1	61·3	37·8	·9
50–	76	57	19	..	75·0	25·0	..
60–	53	43	9	1	81·1	17·0	1·9
70–	21	16	4	1	76·2	19·0	4·8
	390	254	129	7	65·1	33·1	1·8

According to this table it is only in the marriages contracted at the youngest ages that the numbers void by death of the wife are in excess; from 30 upwards the greater mortality is on the side of the husband. The figures in the table represent the combination of all remarriages; but if we distinguish between second and subsequent contracts, the proportions in the former are 64·3 per cent. terminated by death of husband, to 73·0 per cent. in the latter.

Classifying them again, according to the relative ages of husband and wife at the time of marriage, we have the following proportions:—

Remarriages of Widowers per Cent. terminated by			
	Death of Husband.	Death of Wife.	Divorce.
Wives younger than husbands....	67·6	30·8	1·6
Wives whose ages not recorded ..	63·4	34·5	2·1
Wives older than husbands	50·0	50·0	..

It was necessary to include the divorces, but the numbers are too small for argument; and looking at the revolution which has taken place through the establishment of the more expeditious and less expensive Court of Divorce, the prospects of the future cannot be divined through the experience of the past. It has been a popular prejudice that divorce scandals were a peculiar privilege of the aristocracy, but this notion has most probably arisen from the publicity given by the press (and I fear demanded by the public) to trials where the parties concerned are of exalted position. Wealthy and aristocratic suitors, by the aid of eminent counsel, have their sorrows and sins exposed in trials extending occasionally over several days, while all the notoriety enjoyed by persons in a humble sphere is confined to half-a-dozen lines in a newspaper, and of such as these eight or ten have been disposed of in a day. The prejudices against the grant of "issue" policies engendered by the supposed facilities of obtaining divorces have, it is believed, been greatly exaggerated. I have procured from a Parliamentary Blue Book the number of petitions filed in 1863. They were in all 323, and of these the petitions for dissolution of marriage were 255. In the same year there were 230 judgments given, but the return does not distinguish between judicial separations and divorces. If we assume, for argument sake, that they were all divorces *à vinculo,* and compare them with the 3½ millions of existing marriages, the proportion would be 6½ divorces per annum for every 100,000 married couples. The case is here over stated; for I find from another return moved for by Mr. Malins, that from 11th January, 1858, to 21st August, 1860—say two years and a half—only 239 dissolutions of marriage were decreed, or very nearly at the rate of 100 per annum. These numbers can hardly justify the anxiety that has been felt.

I pass from this unpleasant topic to that which is the most important in the present investigation—the respective proportions of those remarriages which result in the birth of children, and of those which are unfruitful. Table E has been drawn up to show these results from observations on 507 remarriages, and here it must be explained that the whole of that number had not terminated by the death of husband or wife, but many of the marriages still existing on 31st December, 1863, were included, limited, however, to those the duration of which had been a minimum of five years. I am aware that it is not a very extraordinary circumstance (as I may show hereafter) for a first child to be born after the lapse of a longer period than five years, but this is more unusual in re-marriages, and that period may, I think, be considered practically safe for the present purpose.

The general result of the 507 marriages it will be seen is roughly stated at *two-thirds* fruitful and *one-third* unfruitful. The exact percentages are 63·51 and 36·49. A want of regularity will be noticed in the proportions at each quinquennial group of ages, and as it might seem inequitable to include in the same category those wives whose ages are not recorded at the time of marriage (as they might be beyond the age of child-bearing, and thus increase the proportion of unfruitful marriages), I have in the final columns obtained the results from those re-marriages only where the wives' ages are on record, which it will be remembered were almost all under 50, and, to make up for deficiency in numbers, have limited the groups to 10 years of age. Here, then, we have an increased percentage of fruitful marriages, viz., 67·7 per cent., and the diminishing fruitfulness according to advancing age is tolerably distinctly exhibited. The chances of issue to a widower re-married at the age of 50 seem to be about evenly balanced; and to a marriage at 70, the chance of having issue appears to be about 25 per cent. I have no instance of a fruitful marriage contracted after the age of 75.

The re-marriages contracted under 50 years of age with their results have been summed up, and also those above that age, the latter being specially the objects of our solicitude in cases of assurance against issue, and the general conclusion that may be drawn is that of these scarcely more than *one-third* will prove fruitful.

In considering the probability of issue by a second marriage, it might seem natural to refer back to observe what had been the result of the former contract, but this test seems to be of comparatively small value, for having searched out 52 cases of un-

TABLE E (*Peerage Families*).—*Remarriages of Widowers—Proportion of Fruitful and Unfruitful Marriages.*

Ages of Husbands.	No. of Marriages Fruitful.	No. of Marriages Unfruitful.	Total.	Per Cent. of Fruitful Marriages.	Per Cent. of Unfruitful Marriages.
24–29	28	4	32	87·52	12·48
30–	50	3	53	94·34	5·66
35–	68	9	77	88·31	11·69
40–	60	28	88	68·19	31·81
45–	50	16	66	75·75	24·25
50–	26	26	52	50·00	50·00
55–	20	33	53	37·73	62·27
60–	10	27	37	27·03	72·97
65–	5	20	25	20·00	80·00
70–	5	14	19	26·31	73·69
75 & upwds.	..	5	5	..	100·00
	322	185	507	63·51	36·49
Under 50	256	60	316	81·00	19·00
50 & upwds.	66	125	191	34·55	65·45
	322	185	507		

WIVES YOUNGER THAN THEIR HUSBANDS.					
No. of Marriages Fruitful.	No. of Marriages Unfruitful.	Total.	Per Cent. of Fruitful Marriages.	Per Cent. of Unfruitful Marriages.	Ages of Husbands.
13	2	15	87·7	13·3	24–29
65	6	71	91·5	8·5	30–39
49	18	67	73·1	26·9	40–49
21	25	46	45·6	54·4	50–59
7	18	25	28·0	72·0	60–69
2	6	8	25·0	75·0	70 & upwds.
157	75	232	67·7	32·3	
127	26	153	83·0	17·0	Under 50
30	49	79	38·0	62·0	50 & upwds.
157	75	232			

fruitful *first* marriages, it appeared that in *two-thirds* the husband actually had issue by the second marriage, while of the remaining one-third which were unfruitful 33 per cent. of the wives were over 40 years of age, and 4 of the husbands exceeded 70 at the second marriage; whence it would seem to follow that the fruitfulness of a marriage depends in a far greater degree upon the lady than upon her husband.

At this point it may be interesting to note the extreme age to which a parent may have attained at the birth of issue. Of husbands, the extreme limit in the cases under observation is 80 years, of which there are two on record. One of these is more than usually peculiar. If the books are to be trusted, this nobleman's first marriage extended over 14 years, and he had no children. He married a second time, at the advanced age of 71, a lady aged 29. During a period of seven years no child was born, but at the expiration of this time appeared a daughter, followed after an interval of two years by a son. The aged parent survived to see his heir pass his seventh birthday. Another exceptional instance may be quoted in the case of a gentleman who enjoyed the luxury of three marriages. By the first, which extended over ten years, he had 3 children (daughters); by the second, 9 children in fifteen years; and by the third, of thirty-two years duration, he had 10 children, the last, a girl, being born 28 years after his third marriage, he being then 78 years of age. There are other instances of births to fathers of the mature ages of 76 and 77.

I regret that the present collection of facts throws but little light on the question of the extreme age at which a wife may become a mother. Finding no record amongst these observations of a lady giving birth to a child after having passed her 49th year, I take the liberty of mentioning, as a matter of passing interest, a few instances which have otherwise come to my knowledge of births to persons of advanced ages.

Dr. Semple, the Physician to the Standard Assurance Company, has very kindly mentioned to me the case of a connexion of his family who gave birth to a child at 56 years of age; and Dr. Saunders, who holds a similar appointment at the City of Glasgow Office, obligingly furnishes me with an instance in his own family of a lady becoming a mother at the age of 54 or 55. Both gentlemen express their belief in the correctness of the ages quoted, but from various causes are not in a position to furnish documentary evidence thereof.

In Beck's *Medical Jurisprudence* is a table showing that out of 10,000 pregnant women in the Manchester Lying-in Hospital, 1 was in her 52nd year, 1 in her 53rd year, and 1 in her 54th year. Here again the ages would require verification. The Swedish statistics, previously referred to, represent that, of 10,000 births, the proportion of mothers above 50 years would be about 2 only.

I can add two instances which admit of no doubt as to the accuracy of the statement of age. The first was a lady, married comparatively late in life, who gave birth to her first child at the age of 51; and the second, a lady married to her second husband —having had a family by both—who completed her 52nd year in January last, whose youngest child is 9 years of age, and who, on the 2nd instant, was stated by her physician to be in an advanced state of pregnancy (about 8 months).

In this case I possess both the physician's report and the certificate of the lady's baptism.

These may, I think, be taken as quite exceptional cases; and for all practical purposes it may be considered that the period of child bearing terminates at 50 years of age.

The number of children born to each re-marriage, together with the number of years duration thereof, is given in Table F for 367 marriages which have terminated by the death of husband or wife. They are subdivided into classes according as the ages of the wives were known or not recorded, and the assumption that the ladies in the latter class were on the average older is borne out, since 190 marriages in this division produced 467 children, while the smaller number of 170 marriages in the first class produced nearly a hundred more children.

The average number of children born to each re-marriage of a widower would, according to these observations, appear to be 2·85. Classifying them according to the relative ages of husbands and wives, the average number of children would be as follows:—

Where the wife is younger than her husband .	3·32
,, older ,, .	2·14
Where the age is not recorded	2·46

Again, distinguishing between second and subsequent marriages, the results are—

To second marriages, average number of children .	2·95
To third ,, ,, ,, .	1·95

It ought, perhaps, to be noted here that still-born births have been excluded from all these observations.

TABLE F (*Peerage Families*).—*Remarriages of Widowers and Number of Children born, distinguishing the Sexes.*

Ages of Husbands.	Wives' Ages younger.					Wives' Ages not recorded.					Wives' Ages older.										Ages of Husbands.
	No. of Marriages.	Duration.		Children Born.		No. of Marriages.	Duration.		Children Born.		No. of Marriages.	Duration.		Children Born.		No. of Marriages.	Duration.		Children Born.		
		Yrs.	mos.	Boys.	Girls.		Yrs.	mos.	Boys.	Girls.		Yrs.	mos.	Boys.	Girls.		Yrs.	mos.	Boys.	Girls.	
24–29	12	344	8	32	30	10	251	11	27	32	3	71	11	7	8	25	668	6	66	70	24–29
30–	22	454	3	57	66	13	240	3	35	37	..	..		..	..	35	694	6	92	103	30–
35–	31	689	6	75	78	26	370	3	43	49	..	..		..	..	57	1059	9	118	127	35–
40–	30	502	9	55	58	29	540	0	34	41	4	78	10	..	..	63	1121	7	89	99	40–
45–	14	278	2	24	33	29	512	7	41	44	..	..		..	..	43	790	9	65	77	45–
50–	14	216	6	12	11	19	261	3	31	18	..	..		..	..	33	477	9	43	29	50–
55–	17	236	9	12	4	22	254	0	12	7	..	..		..	..	39	490	9	24	11	55–
60–	13	138	4	2	6	17	151	0	2	1	..	..		..	..	30	289	4	4	7	60–
65–	9	109	2	5	1	12	114	3	4	3	..	..		..	..	21	223	5	9	4	65–
70–	7	79	2	1	2	9	50	7	1	5	..	..		..	..	16	129	9	2	7	70–
75 & upwds.	1	1	1	..	..	4	10	0	..	..	..	..		..	..	5	11	1	..	..	75 & upwds.
	170	3050	4	275	289	190	2756	1	230	237	7	150	9	7	8	367	5957	2	512	534	
				564					467					15					1046		
Second marriages . . .	155	2859	10	260	268	171	2504	2	213	220	5	113	0	7	8	331	5477	0	480	496	Second marriages . . . .
Third and fourth do.	15	190	6	15	21	19	251	11	17	17	2	37	9	..	..	36	480	2	32	38	Third and fourth do.
	170	3050	4	275	289	190	2756	1	230	237	7	150	9	7	8	367	5957	2	512	534	

Table G is derived from the previous table, and in it will be found the average number of children born according to the ages of the husbands at marriage. To remarriages under 50 years of age, which would probably have the characteristics of first marriages, the average number of children born would appear to be 4·06, while to those contracted after 50, the class in whom we are more especially interested, the average is ·97—say one child only. This is a very important consideration in connexion with the grant of assurances against issue. Mr. Hendrik's *Swedish Statistics* unfortunately do not distinguish between the issue of first and subsequent marriages; but the average number of children born in all the marriages during 40 years was 4·23. It is very satisfactory to notice that the conclusions of Mr. Sadler agree so closely with the results I have deduced. In his *Law of Population*, he states, that the average number of children of second and subsequent marriages is 2·75, and my tables show an average number of 2·85.

Table G contains also columns exhibiting the probability of the birth of a child to a year of marriage, and the average number of years of marriage to a child born.

The proportion of the sexes of the children born as compared with the relative ages of their parents has long been a subject of interesting discussion. The births in England are stated by the Census Commissioners to have been in the proportion of 104,811 boys to 100,000 girls; and Mr. Samuel Brown and others who have studied the question have shown—and their conclusions have met with general acceptance—that in marriages where the husband is older than the wife the number of male births exceeds that of the female, and similarly that when the wife is the elder the girls predominate. I would refer the members of the Institute to Mr. Brown's excellent paper on this subject in Vol. III. of the Journal. In the same volume will be found a contribution of Herr Rath G. Hopf, on the proportions of children born to parents of the Jewish race in Prussia where the male births are to the female in the proportion of 111 to 100. This it is sought to account for by certain physiological considerations which can hardly be discussed here. At a very recent meeting of the Statistical Society, in a debate on Mr. Sargant's paper on the Census of 1861, the question was incidentally mentioned. I quote from the Report in the *Insurance Record*.

"Dr. Webster, in alluding to Mr. Sargant's statement that the returns of the large proportion of males over females arose from imperfect registration, said that might be the case to a certain extent, but as he had paid

TABLE G (*Peerage Families*).—*Remarriages of Husbands (terminated by Death of Husband or Wife) and Issue of such Marriages.*

Age of Husbands at Marriage.	Number of Marriages.	Number of Children born.	Number of Children to each Marriage.	Duration of Marriages. (Number of Years.)	Years of Marriage to each Child born.	Probability of Birth of a Child to a Year of Marriage.	Age of Husbands.
24–29	25	136	5·44	669	4·92	·2032	24–29
30–	35	195	5·57	695	3·57	·2805	30–
35–	57	245	4·30	1060	4·33	·2312	35–
40–	63	188	2·98	1122	5·97	·1676	40–
45–	43	142	3·30	791	5·57	·1795	45–
50–	33	72	2·18	478	6·64	·1506	50–
55–	39	35	·90	491	14·03	·0713	55–
60–	30	11	·37	289	26·27	·0381	60–
65–	21	13	·62	223	17·16	·0583	65–
70–	16	9	·56	130	14·46	·0692	70–
75–	5	0	·00	11	..	..	75–
	367	1046	2·85	5959	5·70	·1755	
Under 50	223	906	4·06	4337	4·79	·2088	Under 50
50 and upwards	144	140	·97	1622	11·58	·0863	50 and upwards
	367	1046	2·85	5959	5·70	·1755	

some attention to the point he thought it was owing to another cause. Mr. Sargant had remarked that there was a larger proportion of males registered in country districts than females, whereas in towns the reverse obtained. He (Dr. Webster) thought this arose from physiological causes. In the country, the residents were much stronger physically than those living in towns; and it was a well-known fact, that if parents were in bad health there was greater chance of their offspring being female; hence it happened that, in the country, the parents being more physically strong than townspeople, the males predominated. This would account, in a great degree, for the disparity which the author of the paper had referred to.

"Dr. Griffiths caused some laughter by saying that he was directly opposed to the principle just laid down. The variation in the respective ages of the parents, and not the strength or health, generally determined the sex."

By a glance at Table F it will be seen at once that in remarriages among the peerage families, the proportions of the sexes of the children differ entirely from those which might have been anticipated from the foregoing theories. Instead of the greater number of children born being boys, the reverse is really the case, and this is the more remarkable, as the difference of age between the husbands and wives must be admitted to be far greater in second than in first marriages, and according to the theory there ought in consequence to be a larger number of male than of female births. Of 1,046 children born, the number of boys appears to have been 512 and of the girls 534, or in the proportion of 95·9 boys to 100 girls, the normal rate of the population being, as before quoted, 105 boys to 100 girls.

Classifying the results in the table we have these proportions:—

Remarriages of widowers with		
Wives younger	95·15	boys born to 100 girls.
Wives whose ages are not on record .	97·05	,, ,,
Wives older	87·50	,, ,,

It does not seem that for all ages of the husband the same results follow, for dividing them into the two classes of husbands under 50, and husbands exceeding that age; we have for the former 430 male to 476 female births, while for the latter the reverse is the result—viz., 82 boys to 58 girls. The numbers are probably too small at the older ages to warrant our accepting these proportions as conclusive, but the general result that second marriages produce a larger proportion of female children, may, I think, be admitted. This conclusion is confirmed by Mr. Sadler, but as his observations were on the same class it remains to be seen whether the same law holds good for the general population and for the families of the aristocracy.

It is a very curious circumstance that although the number of female children born in second marriages exceeds that of the males, the first-born child is, in the majority of instances, a son.

Thus, out of 307 fruitful marriages, the first-born in 168 cases (54·7 per cent.) was a son, and in the remaining 129 (45·3 per cent.) a daughter. I am quite at a loss to account for this apparent inconsistency.

I had proposed to work out the average duration between the date of marriage and the birth of the first child, but upon reflection that the most important feature in the consideration would be the age of the wife (in which my present statistics are comparatively weak), I have deferred, until the investigation respecting first marriages, entering minutely into the question—offering, however, in the mean time a short table to show, out of 100 *first-born* children, the number which have been produced in the first, second, and subsequent years of a marriage.

Percentage of FIRST-BORN *Children in each Year of Marriage.*

	1st Year.	2nd.	3rd.	4th.	5th.	6th.	7th.	8th.	9th.
Boys............	32·4	48·3	11·9	5·3	·7	·7	..	..	·7
Girls............	37·4	44·9	8·4	7·5	..	·9	·9	..	..
Irrespective of sex	34·5	46·9	10·4	6·2	·4	·8	·4	..	·4

From this it would appear that about one-third of the *fruitful* marriages produce children before the end of the first year—that in nearly one-half the first child of the marriage is not born until the end of the first year and before the close of the second—and that not more than one-fifth are without the blessings of offspring beyond two years.

The limits of duration, before the birth of issue, in the re-marriages under observation, are between one month and nine years; both may be considered exceptional cases.

Rather anticipating, in consequence of its importance in regard to issue risks and its exceptional character, I cannot forbear quoting one instance from first marriages of a long-deferred birth:—A lady, aged 18, married to a husband aged 26, gave birth to her first and only child 29½ years after the date of marriage. She must, therefore, have been at the time of the child's birth nearly 48 years of age. Perhaps it should be added that the infant died on the same day.

In bringing these remarks to a close, I am conscious that objections may be urged to the conclusions drawn therein on the

ground of the comparatively small number of re-marriages over which the observations extended. I can only lament that I have not had access to a larger number of facts, and that I am acquainted with no other source from which to draw further trustworthy information.

Although I cannot but admit that at individual ages the results may be open to objection on the ground of insufficiency of numbers, I must be allowed to express some confidence in the general conclusions, and I trust that I may be fairly considered to have added somewhat to the store of knowledge upon which, as Actuaries, we have to draw in estimating the values of our risks under policies of assurance against issue.

On a Problem in Annuities, and on Arbogast's Method of Development. By PROFESSOR DE MORGAN.

IN the investigation of a little curiosity in the matter of annuities, I had occasion to want four or five terms of log $(a+bx+cx^2+\ldots)$, and, by Arbogast's rules I accordingly wrote down the following, without a stroke of the pen more than is here given:—

$$\log a + \frac{b}{a}\cdot x + \left(\frac{c}{a} - \tfrac{1}{2}\frac{b^2}{a^2}\right)x^2 + \left(\frac{d}{a} - \frac{bc}{a^2} + \tfrac{1}{3}\frac{b^3}{a^3}\right)x^3$$

$$+\left(\frac{e}{a} - \frac{bd}{a^2} - \tfrac{1}{2}\frac{c^2}{a^2} + \frac{b^2c}{a^3} - \tfrac{1}{4}\frac{b^4}{a^4}\right)x^4$$

$$+\left(\frac{f}{a} - \frac{be}{a^2} - \frac{cd}{a^2} + \frac{b^2d}{a^3} + \frac{bc^2}{a^3} - \frac{b^3c}{a^4} + \tfrac{1}{5}\frac{b^5}{a^5}\right)x^5 + \ldots$$

This I might have continued to any number of terms for little beyond the trouble of writing. The method is easy to demonstrate and easy to practice; and it struck me that I might, in a few pages, give an account of it sufficient for use and detached from anything else. I therefore append it to the problem.

I have never met with the following little matter, and think it may be offered to the *Journal* for preservation in what an old dedicator called the "storehouse of pretty conceits."

When an annuity is payable at equal intervals in each year, every one sees—or thinks he sees—that the present value is greater than that of a yearly annuity: for, he says, the sooner you are to get money the more is it now worth. He is right: but the supposition being, as usual, that interest is payable as often as the

annuity, it is hardly safe to decide a race between $(1+r)^k$ and $\left(1+\frac{r}{m}\right)^{mk}$ by *à priori* arguments.

Let us call the present value of all the payments made in the kth year by the name of the *kth collection.* Thus in quarterly payments the fifth collection is the sum of the present values of the 17th, 18th, 19th, and 20th payments. Now it is not true that *every* collection is more valuable than the single payment at the end of the year: when a certain time has elapsed, the collections are severally worth less than their single payments. A perpetual annuity is the same however often payment and interest may be subdivided; hence any two annuities, of differently distributed payments, must show a turning point, before which the collections of one exceed in present value, and after which they fall short of, the collections of the other.

Now the curiosity is that the year which contains the turning point does not depend on the number of subdivisions. At $\frac{r}{m}$ per pound per mth of a year, m being the number of payments in a year, and £$\frac{1}{m}$ each payment, the turn has just been made at the end of the $\left(\frac{1}{r}+1\right)$th year, or else the year in which that fraction falls. Thus, at 10 per cent., when $\frac{1}{r}$ is also 10, we see this in the following table:—

Year.	Five payments.		Two payments.		One payment.
1	·9427	>	·9297	>	·9091
9	·4269	>	·4259	>	·4241
10	·3867	>	·3863	>	·3855
11	·3502	<	·3504	<	·3505
12	·3172	<	·3178	<	·3186

The greater the distribution of payments, the greater the value of the year's collection, until 10 years have passed, and then inversely.

The demonstration is as follows:—Equate the value of the kth yearly payment to that of the kth collection of m payments in one year. This gives

$$\frac{1}{(1+r)^k}=\frac{\left(1+\frac{r}{m}\right)^m-1}{r\left(1+\frac{r}{m}\right)^{mk}}$$

or

$$k=\frac{\log\left\{\left(1+\frac{r}{m}\right)^m-1\right\}-\log r}{\log\left(1+\frac{r}{m}\right)^m-\log(1+r)}.$$

Common development gives

$$k=\frac{1}{r}+\frac{3m+1}{4m}+\frac{5(m-1)}{12m^2}r+\dots$$

and this is always less than $\frac{1}{r}+1$ for every value of r less than unity.

I have separately calculated this series one term further when m is infinite, or the annuity *gradual* (payable *momently,* as they say). And I thus find

$$k=\frac{1}{r}+\frac{3}{4}+0.r+\frac{541}{4320}r^2+\dots$$

I now proceed to an account of Arbogast's process. This may be subdivided into a lower part, easily learnt by common algebra, and a higher part, requiring knowledge of the rules of the differential calculus.

A number of successive letters, $a, b, c, d, \dots$ are given, each of which is called the *derivative* of the preceding. The symbol of the derivative may be the letter D: thus, $b=Da$, $c=Db$, &c.; $c=D^2a$, $d=Dc=D^2b=D^3a$, &c.

Every function of these letters has its derivative. When it is a product of powers, this process of derivation is—

> Multiply a power by its exponent, diminish that exponent by a unit, and introduce the next letter once more; and if such introduction increase an exponent already existing, divide by the exponent so increased.

But this rule is to be applied only as follows: to the last letter *in all cases;* to the last but one *only when the last and last but one are consecutive,* a and b, b and c, &c.; to any before the last but one, *never.*

Thus, $a^3b^4c^7$ gives as its derivative $a^3b^4.7c^6d$, from the last letter, c; and $a^3.4b^3c.c^7$ divided by $7+1$ or 8, from the last but one, b, since b and c are consecutive. But $a^3b^4d^7$ gives nothing but $a^3b^4.7d^6e$, since b and d are not consecutive. The following example will give further practice:—

$$D^0b^5=b^5$$
$$D^1b^5=5b^4c$$
$$D^2b^5=5b^4d+10b^3c^2$$
$$D^3b^5=5b^4e+20b^3cd+10b^2c^3$$
$$D^4b^5=5b^4f+20b^3ce+10b^3d^2+30b^2c^2d+5bc^4$$
$$D^5b^5=5b^4g+20b^3cf+20b^3de+30b^2c^2e+30b^2cd^2+20bc^3d+c^5.$$

Observe that if the series of letters come to an end, all terms which would require a letter which is not forthcoming must be erased. Let the letters end at c, which amounts to saying that the whole series is a, b, c, 0, 0, 0, &c.

$$Da^7 = 7a^6b, \quad D^2a^7 = 7a^6c + 21a^5b^2$$
$$D^3a^7 = (7a^6.0 = 0) + 42a^5bc + 35a^4b^3$$
$$D^4a^7 = (42a^5b.0 = 0) + 21a^5c^2 + 105a^4b^2c + 35a^3b^4.$$

We can now write down $(a+b+c+\ldots)^n$ and $(a+bx+cx^2+\ldots)^n$, when n is integer; for

$$(a+b+c+\ldots)^n = a^n + Da^n + D^2a^n + D^3a^n + \ldots;$$

and we also have

$$(a+bx+cx^2+dx^3+\ldots)^n = a^n + Da^n.x + D^2a^n.x^2 + D^3a^n.x^3 + \ldots$$

Thus we can now write down any integer power, developed, without any writing except the result: as

$$(a+bx+cx^2)^4 = a^4 + 4a^3bx + (4a^3c + 6a^2b^2)x^2$$
$$+ (12a^2bc + 4ab^3)x^3 + (6a^2c^2 + 12ab^2c + b^4)x^4$$
$$+ (12abc^2 + 4b^3c)x^5 + (4ac^3 + 6b^2c^2)x^6 + 4bc^3x^7 + c^4x^8.$$

When we form the derivatives of any function of a, the treatment of a, when it is the last letter, or the last but one before b, is simple differentiation, as it has been already in the case of a power of a. And the rule of the last or last but one is to be preserved as before.

Thus we have for derivatives of ϕa as follows:—

$$D\,\phi a = \phi' a.b$$

$$D^2\phi a = \phi' a.c + \frac{\phi'' a}{2}.b^2$$

$$D^3\phi a = \phi' a.d + \frac{\phi'' a}{2}.2bc + \frac{\phi''' a}{2.3}b^3$$

$$D^4\phi a = \phi' a.d + \frac{\phi'' a}{2}(2bd + c^2) + \frac{\phi''' a}{2.3}.3b^2c + \frac{\phi^{\text{iv}} a}{2.3.4}\cdot b^4.$$

A little attention will show the following law:—

$$D^n\phi a = \phi' a.D^{n-1}b + \frac{\phi'' a}{2}D^{n-2}b^2 + \frac{\phi''' a}{2.3}D^{n-3}b^3 + \ldots + \frac{\phi^{(n)} a}{2.3\ldots n}b^n.$$

And the following is the method of developing a function of any polynomial:—

$$\phi(a+bx+cx^2+\ldots) = \phi a + D\phi a.x + D^2\phi a.x^2 + D^3\phi a.x^3 + \ldots.$$

For example—

$$\sqrt{(a+bx+cx^2+\ldots)}=a^{\frac{1}{2}}+\tfrac{1}{2}a^{-\frac{1}{2}}bx+\left(\tfrac{1}{2}a^{-\frac{1}{2}}c-\tfrac{1}{2}.\tfrac{1}{2}a^{-\frac{3}{2}}\frac{b^2}{2}\right)x^2$$
$$+\left(\tfrac{1}{2}a^{-\frac{1}{2}}d-\tfrac{1}{2}.\tfrac{1}{2}.a^{-\frac{3}{2}}.bc+\tfrac{1}{2}.\tfrac{1}{2}.\tfrac{3}{2}a^{-\frac{5}{2}}\frac{b^3}{2.3}\right)x^3$$
$$+\ldots.$$

If we complete the operations as we go on, we find

$$a^{\frac{1}{2}}+\tfrac{1}{2}a^{-\frac{1}{2}}bx+(\tfrac{1}{2}a^{-\frac{1}{2}}c-\tfrac{1}{8}a^{-\frac{3}{2}}b^2)x^2+(\tfrac{1}{2}a^{-\frac{1}{2}}d-\tfrac{1}{4}a^{-\frac{3}{2}}bc+\tfrac{1}{16}a^{-\frac{5}{2}}b^3)x^3$$
$$+(\tfrac{1}{2}a^{-\frac{1}{2}}e-\tfrac{1}{4}a^{-\frac{3}{2}}bd-\tfrac{1}{8}a^{-\frac{3}{2}}c^2+\tfrac{3}{16}a^{-\frac{5}{2}}b^2c-\tfrac{5}{128}a^{-\frac{7}{2}}b^4)x^4,$$

and so on.

The demonstration of all this is as follows:—Take any power of $b+c+d+\ldots$, say the 10th. If we write down every combination of *ten* letters, with or without repetition, and then make every variation of these letters, each such variation is one of the terms of the tenth power required. Thus

$$bb\ ccc\ dddd\ e \text{ takes } \frac{\lfloor 10}{\lfloor 2\ \lfloor 3\ \lfloor 4\ \lfloor 1} \text{ variations;}$$

where $\lfloor n$ signifies $1.2.3\ldots n$. And

$$\frac{\lfloor 10}{\lfloor 2\ \lfloor 3\ \lfloor 4\ \lfloor 1}\, b^2c^3d^4e \text{ is one of the terms required.}$$

This is the common *multinomial theorem* of algebra.

Now the rule above used, employ it on what letter we please, is sure to change one term of the power into another having some one letter *thrown forward,* some one b changed into c, or some one c changed into d, &c. Try it on e in

$$\frac{\lfloor n}{\lfloor p\ \lfloor q\ \lfloor r\ \lfloor s}\, c^pe^qf^rh^s, \text{ which will become}$$

$$\frac{\lfloor n}{\lfloor p\ \lfloor q\ \lfloor r\ \lfloor s}\cdot\frac{q}{r+1}\, c^pe^{q-1}f^{r+1}h^s, \text{ or } \frac{\lfloor n}{\lfloor p\ \lfloor q-1\ \lfloor r+1\ \lfloor s}\, c^pe^{q-1}f^{r+1}h^s,$$

another term. And here an e is changed into f. If, then, we take b^n and use the process again and again *on every letter,* we shall be manufacturing the terms of $(b+c+d+\ldots)^n$ in a very easy way. But we shall produce the same terms over and over and over again. If we confine ourselves to the last letter, and the last but one when the last two are consecutive, we shall produce each term once and once only. This I have now to prove.

Suppose we take any combination of seven, as *cc eee gg*. These letters are in lots of 2 3 2. Throw back the first letter in the last lot, and repeat this process again and again. We cannot fail by this succession to find our way back to *bbbbbbb*. As follows:—

cceeegg	*ccdeeee*	*cccdddd*	*bcccccc*	*bbbbbbc*
cceeefg	*ccddeee*	*ccccddd*	*bbccccc*	*bbbbbbb*
cceeeff	*ccdddee*	*cccccdd*	*bbbcccc*	
cceeeef	*ccdddde*	*ccccccd*	*bbbbccc*	
cceeeee	*ccddddd*	*ccccccc*	*bbbbbcc*	

This rule always leads back to *bbbbbbb*. This rule, then, inverted, will lead from *bbbbbbb* to any combination of seven we please. But though, as the old proverb says, all roads lead to Rome, it takes some management to know how to go from Rome to any other town we want. In this case, however, we have no difficulty: for we want to send a messenger to every town, and therefore we must put a courier on every road, and on every fork to which we come.

The inverse consists of two parts. The inversion of "throw back the first letter of the last lot," is "throw forward the last letter of the *last lot but one*, when the two last letters are consecutive." Except only when the last lot has but one letter: in this case the inverse rule is "throw forward the last letter of the last lot."

In the regressive rule, *aabbbccc* becomes *aabbbbcc*.
In the progressive rule, *aabbbbcc* becomes *aabbbccc*.
In the regressive rule, *aabbbc* becomes *aabbbb*.
In the progressive rule, *aabbbb* becomes *aabbbc*.

Accordingly, the rule for forming all the combinations is—Begin with repetitions of the first letter, and proceed by the progressive rule just obtained, and through every result of it. Thus, to form all the combinations of three out of *b c d e f*, we have

bbb									
bbc									
bbd	*bcc*								
bbe	*bcd*	*ccc*							
bbf	*bce*	*bdd*	*ccd*						
	bcf	*bde*	*cce*	*cdd*					
		bdf	*bee*	*ccf*	*cde*	*ddd*			
			bef		*cdf*	*cee*	*dde*		
			bff			*cef*	*ddf*	*dee*	
						cff		*def*	*eee*
								dff	*eef*
									eff
									fff

Accordingly, by applying the rule of the last or last but one, and furnishing the coefficients by the rule already given, we obtain all the terms of $(b+c+d+\ldots)^n$. When we want $(b+cx+dx^2+\ldots)$, we change a into bx, b into cx, &c.: that is, every progress introduces a new factor, x: accordingly the result of n progressions under the former rule gives the coefficient of x^n in $(b+cx+\ldots)^n$.

As to $\phi(a+bx+cx^2+\ldots)$, let $b+cx+dx^2+\ldots$ be z: then

$$\phi(a+xz)=\phi a+\phi' a.xz+\frac{\phi'' a}{2}x^2z^2+\frac{\phi''' a}{2.3}x^3z^3+\ldots.$$

$$=\phi a+\phi' a.x(b+Db.x+D^2b.x^2+\ldots)+\frac{\phi'' a}{2}x^2(b^2+Db^2.x+D^2b.x^2+\ldots)$$

$$+\frac{\phi''' a}{2.3}x^3(b^3+Db^3.x+D^2b^3.x^2+\ldots)+\ldots.$$

$$=\phi a+\phi' a.bx+\left(\phi' a.Db+\frac{\phi'' a}{2}\cdot b^2\right)x^2$$

$$+\left(\phi' a.D^2b+\frac{\phi'' a}{2}\cdot Db^2+\frac{\phi''' a}{2.3}\cdot b^3\right)x^3$$

$$+\left(\phi' a.D^3b+\frac{\phi'' a}{2}\cdot D^2b^2+\frac{\phi''' a}{2.3}Db^3+\frac{\phi^{\text{iv}} a}{2.3.4}b^4\right)x^4+\ldots$$

$$=\phi a+D\phi a.x+D^2\phi a.x^2+D^3\phi a.x^3+D^4\phi a.x^4+\ldots.$$

by the mode of forming $D^4\phi a$ already shown.

I shall be happy to solve any difficulties which any of your readers may find.

On a Table for the Formation of Logarithms and Anti-Logarithms to Twelve Places. By PETER GRAY, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

PART III.—*On the methods of Computation and Verification of the Table.*

THE twelve-figure table which accompanied Part I. was extracted from a table extending to twenty-four places, which is still in manuscript. Hence the details I am about to give of the methods of construction and verification made use of, will necessarily have reference to the more extensive table.

The table comprises five* columns, related to each other as in the smaller table; and the logarithms, as stated, extend to twenty-

* A definite relation subsists between t, the number of places in the tabulated logarithms, m, the number by which the process is designated, and c, the number of columns which it is necessary to exhibit. The number of tabular entries requisite, irrespective of the auxiliary table, is $t:m$. The first half of these require a column each, while for the second half one additional column suffices—one period of m figures being dropped at the second entry in it, two at the third, and so on. Hence, generally,

$$c=\frac{t}{2m}+1.$$

When $m=3$ this gives, for $t=12$, $c=3$; and for $t=24$, $c=5$.

The property of which advantage is thus taken to restrict the number of columns is an obvious consequence of the relation

$$\log(1+n)=\text{M}(n-\frac{n^2}{2}+\frac{n^3}{3}-\ldots),$$

the first significant figure of the greatest value of n, in Col. V., being in the thirteenth decimal place.

four places, the preliminary periods of ciphers, in the second and following columns, being suppressed, as in the table referred to. The logarithms were all formed true to twenty-seven places, to enable the twenty-fourth to be made true to the nearest unit. The last four columns were constructed by the method of differences; but the first column, which does not admit of the advantageous application of that method, had to be formed otherwise. I shall first describe the formation of Col. I.

The logarithms in Col. I., indices being neglected, are those of 1000 to 1999; and could I have found them, to the extent wanted, in any table of repute, I should not have scrupled to appropriate them. But I know of no such table containing them. The materials within my reach, and which sufficed for the purpose, were Abraham Sharpe's Table of Common Logarithms to sixty-one places. and Wolfram's Table of Naperian Logarithms to forty-eight places.

Sharpe's table is contained in the earlier editions of Hutton, and also in Callet. It is wanting in my copy of Hutton, and I used it as I found it in Callet. This table gives the logarithms of all numbers to 100, and of the primes up to 1,100. From it therefore I got, directly, the logarithms of the primes from 1,000 to 1,100; and also, by addition of their components, the logarithms of all the composites comprised between my limits. In most cases I took the first fifteen figures of the logarithms abstracted, from Hutton's table to twenty places, on account of the superior legibility of this table to that of the corresponding one of Callet.

Wolfram's table is contained in *Vega's Tables** (Leipzig, 1794, folio). It gives the Naperian logarithms (to forty-eight places) of all numbers from 1 to 2,200, and of the primes up to 10,000. By its aid I formed the logarithms of the remaining primes, 119 in number. To obtain the common logarithms from those given by Wolfram, multiplication by the modulus ·43429 . . . was necessary in each case. This operation was much facilitated by the employment of a table, which I formed on purpose, of the first thousand integer multiples of M to thirty-three places.

The one thousand logarithms composing Col. I. being thus formed, there remained the necessity for their verification. For, whatever amount of care I might have exercised—and I had spared no pains to secure accuracy—it would have been too much to expect that in such a number of extensive and independent operations no error should have escaped notice; and besides, I did not

* This work is very valuable, and is also, I believe, very scarce. The copy I possess belonged to Mr. Baily, at the sale of whose library it was bought by Mr. Woollgar, who bequeathed it to me.

know what reliance was to be placed upon the tables I had made use of. Now, there is a theorem which enables us to assign the sum of any number of consecutive logarithms, and I therefore determined, using this theorem, to verify the logarithms I had formed by addition.

Having inserted the logarithms in the column they were to occupy (writing the last nine figures—the nineteenth to the twenty-seventh—of each in pencil, so as to admit of easy correction and curtailment), I added them in groups of ten, and the sums of these again in groups of five, and thus obtained the sums of the successive fifties, ending with 1,049, 1,099, 1,149, 1,199, &c. I then, by the application of the theorem above referred to, determined the sums of the same fifties as follows:—

The theorem is known as "Stirling's Theorem," having been first given by that mathematician in his *Methodus Differentialis*; but it is now to be met with in most treatises on the Differential Calculus and the Calculus of Differences, where it is given as a particular application of a theorem of much greater generality.* As adapted to common logarithms, the theorem is,

$$\log(1.2.3\ldots\overline{x-1}),\ \text{or}\ \log 1+\log 2+\log 3+\ldots+\log(x-1)$$
$$=\log\sqrt{(2\pi)}+(x-\tfrac{1}{2})\log x-\mathrm{M}\left(x-\frac{\mathrm{B}_1}{1.2x}+\frac{\mathrm{B}_3}{3.4x^3}-\frac{\mathrm{B}_5}{5.6x^5}+\ldots\right),$$

where M is, as usual, the modulus, and B_1, B_3, &c., are the *Numbers of Bernoulli*. These Numbers being tabulated, we have the means of carrying the series that multiplies M to any extent that may be requisite. This depends on the values given to x, and on the number of places that are required to be true in the results. For values of x from 1,000 to 2,000, and thirty places true in the results, the following suffices, on putting in the values of B_1, B_3, &c.:—

$$\log\sqrt{(2\pi)}+(x-\tfrac{1}{2})\log x-\mathrm{M}\left(x-\frac{1}{12x}+\frac{1}{360x^3}-\frac{1}{1260x^5}+\frac{1}{1680x^7}\right).$$

In this form and to this extent I applied the theorem between the limits† 1,000, 1,050, 1,100, 1,150, &c., up to 2,000; and the results ought to have agreed with the sums previously obtained by actual addition. In some cases they did so agree, but in most they did not, and I had much trouble in tracing the various discrepancies to their source. I succeeded in removing them all however, and brought the two sets of results into entire accordance as far as the

* See in particular De Morgan's *Differential and Integral Calculus*, p. 312, and Boole's *Calculus of Finite Differences*, pp. 86, 87.

† It is hardly necessary to mention, that in applying the theorem between limits, the constant, $\log\sqrt{(2\pi)}$, disappears.

twenty-sixth decimal place. Some of the discrepancies I found arose from arithmetical errors committed by myself, and the others originated in errors in the tables I had employed. Of these tabular errors, I discovered no fewer than thirteen, namely, seven in Hutton, five in Callet, and one in Wolfram. I give a list of them below.*

But although the correctness of the *sums* of the successive fifties was thus sufficiently established, it by no means followed that the individual values composing them were correct; there might be compensating errors, which computers know are of not particularly rare occurrence. Such errors, therefore, had still to be sought for and eliminated. The process I employed for this purpose (which I need not more particularly describe) was equivalent to the differencing out of the whole column, as far as the differences of the seventh order. In an order of differences so remote, any error in the numbers operated upon would show itself in an intensely aggravated form. In the application of this process no errors were discovered; and hence the accuracy of Col. I. may be considered as satisfactorily established.

The remaining columns were constructed by the method of differences. Fundamental values were obtained by the formation of each hundredth term by the theorem,

$$\log(1+n)=\mathrm{M}\left(n-\frac{n^2}{2}+\frac{n^3}{3}-\ .\ .\ .\ .\ \right).$$

By a first interpolation, nine terms were inserted in each interval of the series thus formed; and by a second, nine terms were again inserted in each interval of the new series. The number of orders of differences it was necessary to make use of in the several interpolations were, in Col. II., seven and five; in Col. III., four and three; in Col. IV., three and two; and in Col. V., two and one.

* The tables in which the errors were found are Hutton's and Callet's tables to twenty places, and Wolfram's table to forty-eight places. The logarithms being in all the tables arranged in periods, it is sufficient to give the erroneous period and its correction.

Hutton.

log 149,	~~1~~3826	should be	03826
1071,	94~~6~~08	„	94708
1085,	8~~51~~48	„	84548
1105,	21~~7~~29	„	21129
1115,	84~~7~~79	„	84179
1125,	47~~9~~81	„	47381
1135,	29~~7~~41	„	29141

Callet.

log 965,	5~~8~~538	should be	56538
1022,	9~~0~~700	„	92700
1082,	7260~~8~~	„	72607
1154,	5808~~7~~	„	58088
1158,	8559~~6~~	„	85593

Wolfram.

log 1409, 16~~6~~6 should be 1696.

As the whole of the above errors may not be found in all the editions of Hutton and Callet, it is proper that I should mention that my copies of these works are of the editions following:—Hutton, 7th edition, 1838; and Callet, "(1795) An 3e." I find also that the two twenty-figure tables referred to, are very far from being universally true to the nearest figure in the twentieth place.

Series formed by interpolation, as just described, need no other verification than that which is afforded in the course of the operation. Hence, in the case of Cols. II. to V., it was only necessary to guard against errors of copying. With this view, after the terms composing these columns were inserted in their places, they were added in groups of ten; and the sums being differenced out as far as was necessary, the regularity in the progression of the last order of differences formed, furnished sufficient assurance of the accuracy of the whole.*

The table hereto appended, as already stated, has been extracted from the larger table whose construction has just been described. After careful reading with the copy, the sums of every ten terms were compared with those formed for verification of the larger table; and I trust it will be found that the precautions thus taken to secure accuracy have been successful.

Before leaving the subject of the twenty-four-figure table, I give an example of its application, being one that involves the use of both the direct and the inverse processes.

Given

$$\Delta^2 \log^{-1}(\cdot 1376) = (10^{\cdot 0001} - 1)^2 10^{\cdot 1376};$$

it is required to evaluate the number thus symbolized.

If we call the number required N, we shall have

$$\begin{aligned} \log \mathrm{N} &= 2\log(10^{\cdot 0001} - 1) + \log 10^{\cdot 1376} \\ &= 2\log(10^{\cdot 0001} - 1) + \cdot 1376. \end{aligned}$$

By the exponential theorem,†

$$a^x - 1 = \mathrm{A}x + \frac{\mathrm{A}^2 x^2}{1.2} + \frac{\mathrm{A}^3 x^3}{1.2.3} + \frac{\mathrm{A}^4 x^4}{1.2.3.4} + \ldots,$$

where A is the Naperian logarithm of a. Making in this $a = 10$, A consequently $= 2\cdot 302585 \ldots$, and $x = \cdot 0001$, we find

$$10^{\cdot 0001} - 1 = \cdot 00023{,}02850{,}20824{,}75268{,}35942{,}55671{,}94133 \ldots.$$

* I carefully preserve these sums, as well as the corresponding sums of Col. I., for use in verifying the several columns, in case the large table should be hereafter put in type.

† There is another theorem that would seem more directly to answer the end in view, namely,

$$(a^x - 1)^n = (\mathrm{A}x)^n + \frac{\Delta^n 0^{n+1}}{1.2 \ldots n+1}(\mathrm{A}x)^{n+1} + \frac{\Delta^n 0^{n+2}}{1.2 \ldots n+2}(\mathrm{A}x)^{n+2} + \ldots,$$

which, on making $n = 2$, and taking the values of $\Delta^2 0^3$, $\Delta^2 0^4$, &c., from a table of the *Differences of Nothing*, becomes

$$(a^x - 1)^2 = (\mathrm{A}x)^2 + (\mathrm{A}x)^3 + \frac{7(\mathrm{A}x)^4}{12} + \frac{(\mathrm{A}x)^5}{4} + \frac{31(\mathrm{A}x)^6}{360} + \frac{(\mathrm{A}x)^7}{40} + \frac{127(\mathrm{A}x)^8}{20160} + \ldots$$

But the exponential theorem is rather more easily used; and as it is the logarithm of $(10^{\cdot 0001} - 1)^2$ that is wanted, the theorem last named answers the purpose just as well.

The rest of the operation, involving the use of the tables, is as follows —

$10^{\cdot 0001} - 1 =$	·0 002 302 850 208 247 526 835 942 557	÷2
	1·151)425 104 123 763 417 971 278	369
	79 80	
	10 744	
	1·151 424 719)385 123 763 4	334
	39 696 347 71	
	5 153 606 147	
	1·151 425 103 575 85$\cancel{6}$)547 907 271 971 278	475
	87 337 230 540 936	
	·6 737 473 290 626	
	1·151 425 104 12$\cancel{3}$)980 347 772 747	851
	59 207 689 449	
	1 636 434 243	
	485 009 139	421
	24 489 097	
	1 410 595	
	259 170	225
	28 885	
	5 856	
	99	086
	7	
	301 029 995 663 981 195 213 739	log 2
	61 075 323 629 791 801 848 963	151 Col. 1
	160 225 104 108 290 795 851	369 „ 2
	145 054 332 731 613 893	334 „ 3
	206 289 878 855 051	475 „ 4
	369 584 604 100	851 „ 5
	182 837 976	421 „ 6
	97 716	225 „ 7
	37	086 „ 8
$\log(10^{\cdot 0001} - 1) =$	$\bar{4}$·362 265 689 658 873 665 867 326	×2
„ $(10^{\cdot 0001} - 1)^2 =$	$\bar{8}$·724 531 379 317 747 331 734 652	
„ $10^{\cdot 1376} =$	0·137 6	
„ N =	$\bar{8}$·862 131 379 317 747 331 734 652	log 6
	778 151 250 383 643 632 508 767	
	83 980 128 934 103 699 225 885	213 Col. 1
	83 860 800 866 572 974 202 474	
	119 328 067 530 725 023 411	274 „ 2
	118 980 388 472 549 269 441	
	347 679 058 175 753 970	

347 679 058 175 753 970	800 Col. 3
347 435 446 548 441 373	
243 611 627 312 597	560 „ 4
243 204 909 797 724	
406 717 514 873	936 „ 5
406 499 635 061	
217 879 812	501 „ 6
217 581 535	
298 277	686 „ 7
297 926	
351	808 „ 8
351	
1·213	274
242 6	
84 91	
4 852	
1·213 332 362	800
970 665 889 6	
1·213 333 332 665 88$\cancel{9}$ 6	560
606 666 666 332 945	
72 799 999 959 953	
1·213 333 333 34$\cancel{6}$ 356 266 292 898	936
1 092 000 000 011	
36 400 000 000	
7 280 000 000	501
606 666 667	
1 213 333	686
728 000	
97 066	
7 280	808
971	
10	
1·213 333 333 346 492 555 006 236	×6
N=0·00 000 007 280 000 000 078 955 330 037 416	

I use short division here in the resolving process, and the figures shown are all that it is necessary to set down. I might have used log 7 instead of log 6 in the decomposition of log N. Had I done so, the process would have been shortened, as Cols. 2

and 3 would have been passed over; but I wished to present the process in its fullest extent.

I came upon the above somewhat remarkable number when constructing a table of $\log^{-1}x$; and being unable, with the means immediately available, to get to the end of the array of ciphers that follow the first three significant figures, I was startled by its seemingly near approach to rationality.

I take this opportunity of presenting a variety of the compounding process, as applied to the above example.

1·000 000 000 000 936 501 686 80$\not{8}$	213
200 000 000 000 187 300 337 362	
10 000 000 000 009 365 016 868	
3 000 000 000 002 809 505 060	
1·213 000 000 001 135 976 54$\not{6}$ 098	274
242 600 000 000 227 195 309	
84 910 000 000 079 518 358	
4 852 000 000 004 543 906	
1·213 332 362 001 136 28$\not{7}$ 803 671	800
970 665 889 600 909 030	
1·213 333 332 667 02$\not{5}$ 888 712 701	560
606 666 666 333 513	
72 799 999 960 021	
1·213 333 333 346 492 555 006 235	$\times 6$
·00 000 007 280 000 000 078 955 330 037 410	

If we were to form, in the usual way, the product of the last four factors (half the total number), we should find that, owing to their peculiar form, it would be a number of the same form, the last half of the decimal portion of which would consist of the significant figures of the several factors in order. We, therefore, in accordance with this remark, form the product of the last four factors by inspection, and continue the compounding process from this point in the usual way. The advantage we should anticipate from this summary mode of using up half the factors is, however, visibly not realized. There are a hundred figures more written in this last form of the operation than in the other. Many of them, it is true, are ciphers, and *might* be omitted; but this would hardly be safe, and on the whole I much prefer the other form, always supposing that paper ruled in squares is used.

In another and concluding paper I shall give a history of the method.

On a Method of Graduation applied to the Peerage Mortality deduced by Mr. Bailey and Mr. Day, with Tables founded thereon. By G. W. BERRIDGE, *Esq., of the London and Provincial Law Assurance Society.*

[Read before the Institute, 24th April, 1865.]

IN a paper read before this Society on the 29th of April, 1861, and contained in the ninth vol. of the *Journal*, there is given a table showing the probability of living a year at each age, as evidenced by the males of the families of the peerage existing within the last half century; and it appeared to me that it would be worth while, if only as a matter of curiosity, to put the information there given into a form better adapted to ordinary calculation.

The first thing to be done was to eliminate the irregularities of the table. It is evident that if the values of $\log p_x$ are graduated whilst their sum remains unchanged, the probability of living over the period dealt with is also unaffected; the original table is, therefore, adhered to the more closely according to the shortness of the periods chosen, and a series of differences will supply the means of graduation over any number of periods—an incidental advantage of this method being that it passes at once from the given to the adjusted values.

By taking the sums of the values of $\log p_x$ in periods of ten years, and redistributing them by means of a fifth difference series, I have constructed a series forming an unbroken curve from age 15 to age 74, approximating very closely to the original data. I dealt with the middle of the table first, as being of most consequence, and added the remaining terms at the beginning and end of the table, in a somewhat different manner. The effect of this process of course is, that the numbers living at the end of each decade are the same as would be shown by a table constructed with the ungraduated probabilities. The series is of a partially geometric character, and is similar to that employed by Dr. Farr in the last English Life Table, the only difference being in the number of orders of differences employed.

In order to obtain the initial values of each order of differences, the values of $\log p_x$ were summed for each decade from 15 to 74 (the characteristic figure being omitted and the mantissæ treated as whole numbers), and differenced to a final quantity. The following are the figures of this operation:—

99616210					
	−18425				
99597785		−9898			
	−28323		−186096		
99569462		−195994		+213315	
	−224317		+27219		−1229911
99345145		−168775		−1016596	
	−393092		−989377		
98952053		−1158152			
	−1551244				
97400809					

The first column contains the sums of $\log p_x$ (unadjusted) for each decade, and the first quantities in each succeeding column are the values required for the next step. These values I call Δ_1, Δ_2, &c.; and by means of the following formulæ the first terms of five orders of differences (δ_1, δ_2, &c.) are obtained, which, with a suitable first term, will produce a series consisting of the adjusted values of the logarithm of the probability of living a year at each age:—

$$\begin{aligned}
\delta_1 &= \cdot 01\Delta_1 - \cdot 009\Delta_2 + \cdot 007725\Delta_3 - \cdot 0066975\Delta_4 + \cdot 005895225\Delta_5 \\
\delta_2 &= \cdot 001\Delta_2 - \cdot 00135\Delta_3 + \cdot 0014625\Delta_4 - \cdot 0014805\Delta_5 \\
\delta_3 &= \cdot 0001\Delta_3 - \cdot 00018\Delta_4 + \cdot 0002355\Delta_5 \\
\delta_4 &= \cdot 00001\Delta_4 - \cdot 0000225\Delta_5 \\
\delta_5 &= \cdot 000001\Delta_5
\end{aligned}$$

These formulæ are, of course, applicable to a series of smaller dimensions, as in such a case the higher differences vanish.

The remaining step is to find the first term ($\log p_{15}$), this equals $\frac{1}{10}\left\{{}_1\Sigma_{10}-\left(\frac{10.9}{1.2}\delta_1+\frac{10.9.8}{1.2.3}\delta_2+\&c.\right)\right\}$, ${}_1\Sigma_{10}$ being 99616210, the first of the quantities used in the preliminary differencing.

The first term of the series and of each order of differences having been obtained, and the series of $\log p_x$ for the ages 15 to 74 constructed from them, there remained the extremities of the table to be added. With regard to the ages prior to 15, the terms being few and extremely irregular, the result, when any method of calculation was employed, varied considerably, according to the grouping. I therefore abandoned any such attempt; set out the logarithms of p_x from age 3 to age 15 as ordinates to a curve, and then drew a curve through them corresponding as closely as possible to the original quantities and their sums from age to age, without making any sudden turn. In translating this curve into figures, I followed the same condition of making the sums of the logarithms equal in the adjusted and unadjusted tables. I was obliged to alter the value at age 15 slightly.

The remaining values of $\log p_x$ for the 18 years from 75 to 93, were divided into two groups of 9 each, and then summed and redistributed in a third difference series; but in order to effect a junction with the preceding series, its last term, and the last term

of the first order of differences, were introduced into the calculation in the following manner:—Making these values equal to z and ${}_w\delta_1$, we have ${}_w\delta_1+d_2$ and d_2+d_3 to take the place of the first and second differences to a series beginning with z; and the required new quantities d_2 and d_3 are easily found, having z and the sums of the two groups given.

$$\begin{array}{cccccc} z & & {}_w\delta_1 & & d_2 & \\ & {}_w\delta_1+d_2 & & d_2+d_3 & & d_3 \end{array}$$

Calling these two sums A and B, we have

$$A+z=10z+\frac{10.9}{1.2}({}_w\delta_1+d_2)+\frac{10.9.8}{1.2.3}(d_2+d_3)+\frac{10.9.8.7}{1.2.3.4}d_3$$

$$A+B+z=19z+\frac{19.18}{1.2}({}_w\delta_1+d_2)+\frac{19.18.17}{1.2.3}(d_2+d_3)+\frac{19.18.17.16}{1.2.3.4}d_3;$$

d_2 and d_3 being found from these two equations, their continuous addition to the final quantities in the preceding series and its first order of differences, will give the values of $\log p_x$ required to complete the table. The table ends rather abruptly at age 93, but so do the original data.

The values of $\log p_x$ being thus obtained, they were easily checked by addition, as the sum of each group would, if correct, agree with the numbers on which they were based; and from them I have constructed a table containing the numbers living at each age and their decrements, the expectation of life, columns D and N, and the values of annuities—columns D and N being constructed in the ordinary manner, and the annuity column by Mr. Gray's method, so as to check one another.

It will be objected that this table is unsuited for many purposes, on account of the decrease in the probability of dying in a year from age 22 to age 34; and to illustrate the extent to which this goes, I have calculated a table of the annual premiums for a term insurance for 7 years for the 20 years from age 15 to 24. The premium at age 20 is very high, it then decreases to age 31, when it begins to increase again.

Premium per Cent. for Term Assurance for Seven Years.

Age 15	·780	Age 25	·928
„ 16	·864	„ 26	·899
„ 17	·941	„ 27	·879
„ 18	·988	„ 28	·862
„ 19	1·012	„ 29	·850
„ 20	1·023	„ 30	·842
„ 21	1·012	„ 31	·835
„ 22	1·001	„ 32	·844
„ 23	·979	„ 33	·855
„ 24	·952	„ 34	.870

In order to show that this is justified by the original data, I have set out the curve formed by the natural numbers of the probability of living a year at each age, adjusted and unadjusted. From this it will be seen that the mortality from age 21 to age 26 is not by any means exaggerated. The dip in the line expressing the original values of p_x extends from age 14 to age 36, and the number of deaths recorded between these ages is 375, out of a total of 1,938. But even if we only take the deaths at those ages which show a greater mortality than age 34, the turning point of the new table, we get a total of 168. I think, therefore, that a climacteric extending over so many years, and defined by so large a proportion of the mortality, is not to be disregarded, and that any graduation which should attempt to erase it would be fallacious. Another peculiarity of the table is the rapid increase in the rate of mortality after age 80, which is also, I think, borne out by the data. From the diagram it will be seen that the amount of deviation of the new quantities from the data is very small—smaller, perhaps, than what is required in order to render the table available for practical purposes; but this could not be done without effacing the peculiarities of the table.

In one of the annexed tables a comparison is instituted between the annuities derived from the peerage mortality and those given in some tables in general acceptation. The result of the comparison may, perhaps, be stated generally in the following manner. The peerage annuities are higher than the others at all the earlier and middle ages, with the exception of ages 10 and 15 in Davis's Equitable, and 15 and 20 in the Carlisle Experience and Equitable tables (Morgan's); falling below them at the older ages, in the following order—Davis's Equitable at age 60, Carlisle at 65, Morgan's Equitable at 70, English Life No. 1 at 75, English Life No. 3 and Experience at 80. The greatest difference in the middle of life is that between the peerage and English Life No. 3; it amounts to rather more than one year's purchase. These remarks apply, of course, inversely to the premiums—the maximum difference in the annual premiums being at age 50, and amounting to 9*s*. 8*d*. per cent.

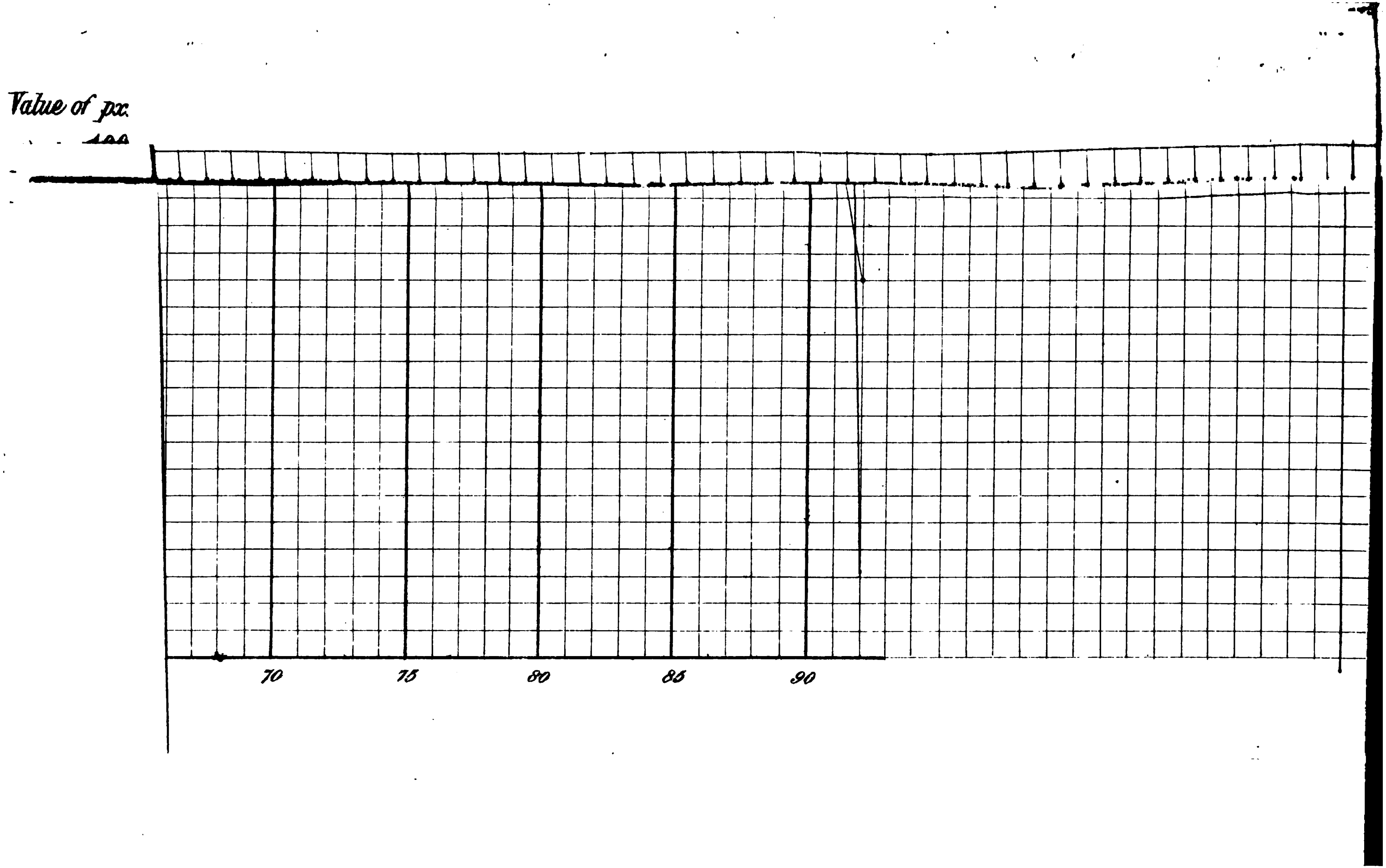
Another interesting question respecting any new table of mortality is the effect it would have if used in a valuation. In order somewhat to elucidate this point, I have calculated the values of a policy at intervals of five years, the age at entry being taken at each ten years from 20 to 70, and compared them with the Carlisle values taken from Mr. Hall Todd's tables, and also with the

values obtained on the supposition that Carlisle premiums are charged and the peerage table used in the valuation. From these quantities and their totals it will be seen that the values belonging to age 20 are lower in the peerage table than in the Carlisle; but at the other ages they are generally higher, their total being considerably above the Carlisle total, although the annual premiums are lower at most ages. This, of course, arises from the fact that the risk from year to year through the middle of life—say from age 30 to age 55, is comparatively small, and is but a particular way of showing its effect. It also corresponds to a certain extent in its result to that which would arise from the use of a table based on the supposition that a low rate of mortality is to be expected during the earlier years of a policy, which is so commonly experienced and ascribed to the selection of the lives.

The third series of values derived from the valuation of the Carlisle pure premium by the peerage table falls between the other two, being lower than the Carlisle for the younger ages and shorter terms, and higher for the others. It is also higher in the total.

It was, I believe, with the intention of elucidating any peculiarities there might be in the mortality of a particular class that the original investigation was undertaken; and if I have succeeded in giving a monetary value to the conclusions then arrived at, and rendered them applicable to particular cases, I do not think the result is of insufficient value to compensate for the work bestowed on it, even although it may have but a limited application.

[NOTE.—Our readers will probably be reminded by this essay of an ingenious paper by Mr. William Spens, of Glasgow, in which that gentleman argues the question whether there is a materially greater risk in the assurance of a select life of from 40 to 45 than of a select life of from 20 to 25 for one year. The sum of the probabilities given in Mr. Berridge's Table I. for the former period is ·49461, and for the latter ·49469, the two being all but identical; and it is possible that in almost every mortality table some similar anomalies will be found. The probabilities of living, for instance, in the Carlisle table from age 46 to age 50, and from age 91 to age 96, increase instead of diminishing, to say nothing of more minute aberrations. But it may be argued that these facts serve only to show that amongst the lives observed there is, in every case, more or less unsoundness; and that they do not controvert the accuracy of the theory that, in the absence of disturbing causes, the declension in vitality from early youth, if not from birth, till death would be in a progression uniformly accelerated. Mr. Spens will, we dare say, readily concede that selection as ordinarily practised does not eliminate all unsoundness.—ED. *A. M.*]



Value of px.
100
70
75
80
85
90

TABLE I.—*Probability of Surviving a Year (Peerage, Males).*

Age.	Original Values.	Graduated Values.	Age.	Original Values.	Graduated Values.
0	·92179	·92179	47	·98686	·98656
1	·98375	·98375	48	·98647	·98597
2	·99135	·99135	49	·98761	·98538
3	·99610	·99518	50	·98433	·98478
4	·99564	·99565	51	·98618	·98418
5	·99519	·99610	52	·98814	·98355
6	·99824	·99635	53	·98289	·98290
7	·99734	·99658	54	·97732	·98221
8	·99555	·99674	55	·98344	·98148
9	·99642	·99684	56	·97839	·98068
10	·99596	·99674	57	·98963	·97979
11	·99414	·99660	58	·97367	·97878
12	·99639	·99641	59	·97959	·97764
13	·99504	·99619	60	·97999	·97631
14	·99774	·99582	61	·97182	·97477
15	·99592	·99527	62	·97283	·97297
16	·99275	·99436	63	·96236	·97086
17	·99362	·99257	64	·97014	·96839
18	·99267	·99124	65	·95687	·96550
19	·99217	·99030	66	·96242	·96213
20	·99163	·98969	67	·96017	·95821
21	·98344	·98935	68	·94662	·95366
22	·99035	·98921	69	·94108	·94841
23	·98837	·98924	70	·93461	·94238
24	·99115	·98939	71	·93670	·93548
25	·98861	·98963	72	·94595	·92763
26	·98745	·98991	73	·92543	·91873
27	·99240	·99022	74	·91048	·90871
28	·99132	·99053	75	·88873	·90142
29	·99070	·99081	76	·90646	·89587
30	·98750	·99106	77	·90217	·89110
31	·99415	·99127	78	·88312	·88619
32	·98982	·99141	79	·89499	·88022
33	·99243	·99149	80	·89189	·87231
34	·99346	·99150	81	·82154	·86162
35	·99451	·99144	82	·87500	·84739
36	·98610	·99131	83	·80465	·82892
37	·99371	·99111	84	·77647	·80568
38	·99013	·99085	85	·76923	·77729
39	·99121	·99052	86	·72631	·74356
40	·98633	·99015	87	·66154	·70456
41	·99024	·98973	88	·58974	·66060
42	·98636	·98926	89	·70000	·61225
43	·98928	·98876	90	·57143	·56033
44	·99353	·98824	91	·75000	·50585
45	·98550	·98769	92	·33334	·45001
46	·98509	·98713			

TABLE II.—*Peerage Mortality.* (*Interest 3 per Cent.*)

Age.	Numbers Living.	Decrements.	Expectation of Life.	D.	N.	Value of Annuity.
0	100,000	7821	52·03	100000	2295946	22·959
1	92,179	1498	55·42	89494·2	2206452	24·655
2	90,681	784	55·31	85475·6	2120976	24·814
3	89,897	433	54·79	82268·5	2038708	24·781
4	89,464	390	54·06	79487·6	1959220	24·648
5	89,074	347	53·28	76836·0	1882384	24·499
6	88,727	324	52·49	74307·5	1808077	24·332
7	88,403	303	51·67	71879·7	1736197	24·154
8	88,100	288	50·85	69547·0	1666650	23·964
9	87,812	278	50·02	67300·6	1599350	23·764
10	87,534	285	49·17	65133·5	1534216	23·555
1	87,249	297	48·33	63030·6	1471185	23·341
2	86,952	312	47·51	60986·4	1410199	23·123
3	86,640	330	46·67	58997·6	1351201	22·903
4	86,310	361	45·84	57061·1	1294140	22·680
15	85,949	407	45·04	55167·4	1238973	22·458
6	85,542	483	44·24	53307·0	1185666	22·242
7	85,059	632	43·49	51462·1	1134204	22·040
8	84,427	740	42·83	49591·9	1084612	21·871
9	83,687	811	42·19	47725·5	1036887	21·726
20	82,876	855	41·61	45886·5	991000	21·597
1	82,021	873	41·02	44090·3	946910	21·477
2	81,148	876	40·46	42350·5	904559	21·359
3	80,272	863	39·91	40673·2	863886	21·240
4	79,409	843	39·33	39064·0	824822	21·115
25	78,566	815	38·74	37523·6	787298	20·981
6	77,751	784	38·14	36052·7	751246	20·837
7	76,967	753	37·52	34649·7	716596	20·681
8	76,214	722	36·90	33311·4	683285	20·512
9	75,492	694	36·25	32034·8	651250	20·329
30	74,798	668	35·58	30815·8	620434	20·134
1	74,130	648	34·89	29651·1	590783	19·924
2	73,482	631	34·20	28535·8	562247	19·703
3	72,851	620	33·48	27466·8	534780	19·470
4	72,231	614	32·78	26439·8	508341	19·226
35	71,617	613	32·04	25451·5	482889	18·973
6	71,004	618	31·32	24498·7	458390	18·711
7	70,386	625	30·59	23578·1	434812	18·441
8	69,761	639	29·86	22688·1	412124	18·165
9	69,122	655	29·14	21825·5	390299	17·883
40	68,467	675	28·40	20989·0	369310	17·595
1	67,792	696	27·68	20176·8	349133	17·304
2	67,096	721	26·97	19388·0	329745	17·008
3	66,375	745	26·24	18621·0	311124	16·708
4	65,630	772	25·55	17875·8	293248	16·405
45	64,858	799	24·84	17151·0	276097	16·098
6	64,059	824	24·14	16446·3	259651	15·788

TABLE II. (continued).

Age.	Numbers Living.	Decrements.	Expectation of Life.	D.	N.	Value of Annuity.
47	63,235	850	23·46	15761·9	243889	15·473
8	62,385	875	22·77	15097·1	228792	15·155
9	61,510	899	22·09	14451·8	214340	14·831
50	60,611	922	21·40	13825·8	200514	14·503
1	59,689	945	20·71	13218·9	187295	14·169
2	58,744	966	20·05	12630·7	174665	13·829
3	57,778	988	19·38	12061·2	162603	13·482
4	56,790	1010	18·71	11509·6	151094	13·128
55	55,780	1033	18·04	10975·7	140118	12·766
6	54,747	1058	17·37	10458·7	129660	12·397
7	53,689	1085	16·70	9957·8	119702	12·021
8	52,604	1116	16·06	9472·4	110229	11·637
9	51,488	1152	15·37	9001·4	101228	11·246
60	50,336	1192	14·71	8543·7	92684·2	10·848
1	49,144	1240	14·06	8098·4	84585·8	10·445
2	47,904	1295	13·40	7664·1	76921·7	10·037
3	46,609	1358	12·77	7239·8	69681·9	9·625
4	45,251	1430	12·13	6824·1	62857·8	9·211
65	43,821	1512	11·51	6416·0	56441·8	8·797
6	42,309	1602	10·91	6014·2	50427·6	8·385
7	40,707	1701	10·32	5617·9	44809·7	7·976
8	39,006	1808	9·75	5226·4	39583·4	7·574
9	37,198	1919	9·19	4839·0	34744·4	7·180
70	35,279	2033	8·67	4455·6	30288·8	6·798
1	33,246	2145	8·17	4076·6	26212·2	6·430
2	31,101	2251	7·69	3702·5	22509·7	6·080
3	28,850	2345	7·26	3334·5	19175·2	5·751
4	26,505	2419	6·86	2974·2	16201·0	5·447
75	24,086	2375	6·50	2624·1	13576·9	5·174
6	21,711	2260	6·15	2296·4	11280·5	4·912
7	19,451	2119	5·81	1997·4	9283·1	4·647
8	17,332	1972	5·46	1728·0	7555·0	4·372
9	15,360	1840	5·09	1486·8	6068·3	4·081
80	13,520	1726	4·72	1270·6	4797·7	3·776
1	11,794	1632	4·33	1076·1	3721·6	3·458
2	10,162	1551	3·95	900·17	2821·4	3·134
3	8,611	1473	3·57	740·57	2080·9	2·810
4	7,138	1387	3·20	596·00	1484·9	2·491
85	5,751	1281	2·86	466·21	1018·7	2·185
6	4,470	1146	2·53	351·81	666·84	1·895
7	3,324	982	2·23	253·99	412·85	1·625
8	2,342	795	1·96	173·74	239·10	1·376
9	1,547	600	1·71	111·42	127·68	1·146
90	947	416	1·47	66·22	61·46	·928
1	531	263	1·23	36·05	25·41	·705
2	268	147	·95	17·66	7·74	·438
3	121	121	..	7·74		

TABLE III.—*Comparison of the Values of Annuities according to the Peerage with those of other Tables.*

Age.	Peerage. Males.	Equitable. Davis.	Carlisle.	Equitable. Morgan.	English Life, No. 1. Males.	English Life, No. 3. Males.	Experience.
0	22·96	..	17·32	..	18·22	18·15	..
5	24·50	..	23·69	..	23·33	23·44	..
10	23·56	23·57	23·51	23·30	23·03	23·11	23·36
15	22·46	22·55	22·58	22·63	22·09	22·11	22·63
20	21·60	21·53	21·69	21·86	21·18	21·06	21·80
25	20·98	20·47	20·67	20·89	20·19	20·09	20·84
30	20·13	19·37	19·56	19·77	19·13	19·01	19 75
35	18·97	18·23	18·43	18·52	17·98	17·81	18·52
40	17·60	16·99	17·14	17·15	16·72	16·47	17·12
45	16·10	15·58	15·86	15·63	15·32	15·01	15·54
50	14·50	14·02	14·30	13·91	13·74	13·42	13·82
55	12·77	12·49	12·41	12·09	11·91	11·76	12·02
60	10·85	10·94	10·49	10·27	10·06	10·02	10·19
65	8·80	9·29	8·92	8·49	8·29	8·27	8·38
70	6·80	7·62	7·12	6·80	6·65	6·61	6·69
75	5·17	5·96	5·51	5·26	5·18	5·14	5·15
80	3·78	4·29	4·37	3·78	3·92	3·92	3·80
85	2·19	2·93	3·23	2·63	2·88	2·94	2·62
90	·93	2·01	2·50	1·89	2·05	2·18	1·52

TABLE IV.—*Peerage Mortality. Annual Premiums.* (3 *per Cent.*)

Age.	Premium.	Age.	Premium.	Age.	Premium.	Age.	Premium.
0	1·261	25	1·637	50	3·538	75	13·287
1	·986	6	1·667	1	3·679	6	14·002
2	·961	7	1·700	2	3·831	7	14·796
3	·966	8	1·736	3	3·993	8	15·702
4	·986	9	1·776	4	4·166	9	16·768
5	1·009	30	1·819	55	4·352	80	18·026
6	1·035	1	1·867	6	4·552	1	19·519
7	1·063	2	1·917	7	4·767	2	21·278
8	1·093	3	1·973	8	5·001	3	23·334
9	1·126	4	2·032	9	5·254	4	25·733
10	1.160	35	2·094	60	5·527	85	28·486
1	1·196	6	2·161	1	5·825	6	31·628
2	1·232	7	2·231	2	6·147	7	35·181
3	1·271	8	2·306	3	6·500	8	39·176
4	1·310	9	2·383	4	6·881	9	43·685
15	1·350	40	2·466	65	7·295	90	48·954
6	1·390	1	2·551	6	7·743	1	55·731
7	1·428	2	2·641	7	8·228	2	66·627
8	1·460	3	2·735	8	8·750		
9	1·488	4	2·833	9	9·312		
20	1·513	45	2·936	70	9·911		
1	1·537	6	3·044	1	10·546		
2	1·560	7	3·158	2	11·212		
3	1·584	8	3·278	3	11·900		
4	1·609	9	3·405	4	12·598		

TABLE V.—*Value of £100 Policy, Premium just Due.* (*Interest 3 per Cent.*)

	Duration of Policy.	Age at Entry.						Total.
		20.	30.	40.	50.	60.	70.	
According to Peerage Table.	5	2·725	5·494	8·050	11·205	17·312	20·825	65·611
	10	6·475	12·013	16·628	23·576	34·184	38·754	131·630
	15	11·613	19·096	25·970	36·807	47·890	..	141·376
	20	17·709	26·643	36·284	49·701	59·690	..	190·027
	25	24·334	34·863	47·315	60·176	..	..	166·688
	30	31·393	43·938	58·065	69·194	..	..	202·590
	35	39·080	53·644	66·708	..	..	..	159·432
	40	47·568	63·102	74·316	..	..	..	184·986
	Total	180·897	258·793	333·336	250·659	159·076	59·579	1242·340
According to Carlisle Table.	5	4·534	5·464	7·053	12·374	13·698	19·835	62·958
	10	9·422	11·746	15·655	24·904	29·310	33·958	124·995
	15	14·371	17·970	26·092	35·190	43·332	..	136·955
	20	20·061	25·562	36·660	46·914	53·315	..	182·512
	25	25·699	34·773	45·336	57·445	..	..	163·253
	30	32·575	44·100	55·226	64·941	..	..	196·842
	35	40·919	51·757	64·106	..	..	..	156·782
	40	49·366	60·484	70·430	..	..	..	180·280
	Total	196·947	251·856	320·558	241·768	189·655	53·793	1204·577
Valuation of Carlisle Premiums by Peerage Table.	5	3·147	2·841	5·756	10·040	14·746	23·995	60·525
	10	6·879	9·543	14·547	22·573	32·142	41·206	126·890
	15	11·995	16·826	24·123	35·977	46·272	..	135·193
	20	18·066	24·584	34·694	49·041	58·439	..	184·824
	25	24·662	33·035	46·000	59·652	..	..	163·349
	30	31·690	42·364	57·019	68·789	..	..	199·862
	35	39·344	52·342	65·969	..	..	..	157·655
	40	47·795	62·067	73·675	..	..	..	183·537
	Total	183·578	243·602	321·783	246·072	151·599	65·201	1211·835

A Budget of Paradoxes. *By* PROFESSOR DE MORGAN.

(Continued from p. 108.)

No. XIV. 1830—1833.

1830. The celebrated interminable fraction 3·14159..., which the mathematician calls π, is the ratio of the circumference to the diameter. But it is thousands of things besides. It is constantly turning up in mathematics: and if arithmetic and algebra had been studied without geometry, π must have come in somehow, though at what stage or under what name must have depended upon the casualties of algebraical invention. As it is, our trigonometry being founded on the circle, π first appears as the ratio stated. If, for instance, a deep study of probable fluctuation from the average had preceded geometry, π might have emerged as a number perfectly indispensable in such problems as—What is the chance of the number of aces lying between a million $+x$ and a million $-x$, when six million of throws are made with a die? I have not gone into any detail of all those cases in which the paradoxer finds out, by his unassisted acumen, that results of mathematical investigation *cannot be:* in fact, this discovery is only an accompaniment, though a necessary one, of his paradoxical statement of that which *must be.* Logicians are beginning to see that the notion of *horse* is inseparably connected with that of *non-horse:* that the first without the second would be no notion at all. And it is clear that the positive affirmation of that which contradicts mathematical demonstration cannot but be accompanied by a declaration, mostly overtly made, that demonstration is false. If the mathematicians were interested in punishing this indiscretion, he could make his denier ridiculous by inventing asserted results which would completely take him in.

More than thirty years ago I had a friend, now long gone, who was a mathematician, but not of the higher branches: he was, *inter alia,* thoroughly up in all that relates to mortality, life assurance, &c. One day, explaining to him how it should be ascertained what the chance is of the survivors of a large number of persons now alive lying between given limits of number at the end of a certain time, I came, of course, upon the introduction of π, which I could only describe as the ratio of the circumference of a circle to its diameter. "Oh, my dear friend! that must be a delusion; what can the circle have to do with the numbers alive at the end of a given time?"—"I cannot demonstrate it to you;

but it is demonstrated."—"Oh! stuff! I think you can prove anything with your differential calculus: figment, depend upon it." I said no more; but, a few days afterwards, I went to him and very gravely told him that I had discovered the law of human mortality in the Carlisle table, of which he thought very highly. I told him that the law was involved in this circumstance. Take the table of expectation of life, choose any age, take its expectation and make the nearest integer a new age, do the same with that, and so on; begin at what age you like, you are sure to end at the place where the age past is equal, or most nearly equal, to the expectation to come. "You don't mean that this always happens?" —"Try it." He did try, again and again; and found it as I said. "This is, indeed, a curious thing; this *is* a discovery." I might have sent him about trumpeting the law of life: but I contented myself with informing him that the same thing would happen with any table whatsoever in which the first column goes up and the second goes down; and that if a proficient in the higher mathematics chose to palm a figment upon him, he could do without the circle: *à corsaire, corsaire et demi*, the French proverb says.

The first book of Euclid's Elements. With alterations and familiar notes. Being an attempt to get rid of axioms altogether; and to establish the theory of parallel lines, without the introduction of any principle not common to other parts of the elements. By a member of the University of Cambridge. Third edition. In usum serenissimæ filiolæ. London, 1830.

The author was Lieut.-Col. (now General) Perronet Thompson, the author of the *Catechism on the Corn Laws.* I reviewed the fourth edition—which had the name of *Geometry without Axioms*, 1833—in the quarterly *Journal of Education* for January, 1834. Colonel Thompson, who then was a contributor to—if not editor of—the *Westminster Review*, replied in an article the authorship of which could not be mistaken.

Some more attempts upon the problem, by the same author, will be found in the sequel. They are all of acute and legitimate speculation; but they do not conquer the difficulty in the manner demanded by the conditions of the problem.

Morning Post, Wednesday, May 4, 1831.

"We understand that although, owing to circumstances with which the public are not concerned, Mr. Goulburn declined becoming a candidate for University honours, that his scientific attainments are far from inconsiderable. He is well known to be the author of an essay in the Philosophical Transactions on the accurate rectification of a circular arc, and of an investigation of the equation of a lunar caustic—a problem likely to become of great use in nautical astronomy."

This hoax—which would probably have succeeded with any journal—was palmed upon the *Morning Post*, which supported Mr. Goulburn, by some Cambridge wags who supported Mr. Lubbock, the other candidate for the University of Cambridge. Putting on the usual concealment, I may say that I always suspected Dr-nkw-t-r B-th-n-e of having a share in the matter. The skill of the hoax lies in avoiding the words "quadrature of the circle," which all know, and speaking of "the accurate rectification of a circular arc," which all do not know, for its synonyme. The *Morning Post* next day gave a reproof to hoaxers in general, without referring to any particular case. It must be added that although there are *caustics* in mathematics, there is no *lunar* caustic.

So far as Mr. Goulburn was concerned, the above was poetic justice. He was the minister who, in the old time, told a deputation from the Astronomical Society that the Government "did not care twopence for all the science in the country." There may be some still alive who remember this: I heard it from more than one of those who were present, and are now gone. Matters are much changed. I was thirty years in office at the Astronomical Society; and, to my certain knowledge, every Government of that period, Whig and Tory, showed itself ready to help with influence when wanted, and with money whenever there was an answer for the House of Commons.

(*To be continued.*)

THINGS WORTH NOTING.

3.—SOMETHING MORE ABOUT DE MOIVRE'S FORMULA.

Mr. Baron Maseres, in the preface to his cumbrous work on the *Principles of the Doctrine of Life Annuities* (1783, quarto), at p. xi., after referring to and commending for its utility the method for finding the value of an annuity on (x) in terms of that of an annuity on $(x+1)$, has the following:—

"This method was first communicated to me by Dr. Price, but it was published in the year 1779, by Mr. William Morgan, the actuary to the Society for Equitable Assurances, near Blackfriars Bridge, in his *Treatise on the Doctrine of Annuities and Assurances on Lives*, pp. 56, 57; and it had been published before by Dr. Price himself, in his *Treatise on Reversionary Payments*, Note O of the Appendix, and likewise by Mr. Thomas Simpson, in his book on *Life Annuities*, Prob. 1, Coroll. 7; which last book was published so long ago as the year 1742. But I should suspect that it was not known to Mr. De Moivre when he calculated his tables of the values

of life annuities; for, if it had, I should imagine he would hardly have thought it necessary to have recourse to a certain inaccurate hypothesis concerning the probabilities of life, in order to diminish the labour of his computations, which would have been almost equally facilitated by the use of this excellent method."

Could there be a more pregnant illustration than we have here of the danger of rash generalization? The learned Judge infers that, because De Moivre invented an hypothesis which enabled him to assign with considerable facility the value of an annuity on a life of any age, *therefore* the method referred to in the foregoing extract could not have been known to him. Why, not only was the method *known* to De Moivre, but it was actually *discovered* by him! And more than this, he *appears* (I say *appears*, because I desire to take warning from the example of the Baron) to have *abandoned* it in favour of his hypothesis.*

P. GRAY.

Camden Town, 8th June, 1865.

CORRESPONDENCE.

ON THE CALCULATION OF PREMIUMS RETURNABLE AT DEATH OR WITHDRAWAL.

To the Editor of the Assurance Magazine.

SIR,—On looking over the pages of the last Number of the *Journal* I was struck with a paragraph in a letter signed J. W. Stephenson, wherein the writer professes to give a method of finding the single premium for a certain contingent benefit, with the condition that the premium shall be returnable (without interest) at death, and also in the event of the purchaser wishing to withdraw at any time before the benefit becomes payable.

Now, the determination of the premium required for the assurance of a given benefit, with the return of the premium *at death*, is a very simple matter; the latter contingency being perfectly susceptible of calculation. But as the contingency of having to pay a given sum on *withdrawal* (other than the surrender value of the policy) is not so, I was not a little curious to see how such a problem would be dealt with. The particular benefit discussed by Mr. Stephenson is a deferred annuity, and the following is, substantially, the reasoning by which he arrives at his solution.

Let P_x denote the single premium required; and suppose A, the intending purchaser, deposits this amount at interest in the hands of B, to be held at A's disposal until the time arrives at which the annuity is wanted—say at the expiration of n years. Let the yearly rate of interest which B allows A on his deposit be i per £1, which must also be the rate of interest assumed in the calculation.

With this yearly interest, amounting to $P_x i$, A is enabled to assure a deferred annuity (with forfeiture of premiums in the event of death) of $P_x i \cdot \dfrac{N_x - N_{x+n}}{N_{x+n}}$ per annum; and at the expiration of the period of n years

* See p. 176.

he can withdraw his deposit from B's hands, and with it purchase an immediate annuity of $P_x \cdot \frac{D_{x+n}}{N_{x+n}}$. The whole annuity thus acquired is $P_x \cdot \frac{(N_x - N_{x+n})i + D_{x+n}}{N_{x+n}}$, and equating to unity, the amount to be deposited in order to secure an annuity of £1 is expressed by the equation $P_x = \frac{N_{x+n}}{(N_x - N_{x+n})i + D_{x+n}}$, which is Mr. Stephenson's formula.

By the arrangement here supposed it is evident that in the event of A's death, before the expiration of n years, his representatives will receive from B (at the end of the year of death) the sum of $P_x(1+i)$, or P_x with one year's interest upon it. Now, the object of this letter is to show that the return of this sum in the event of death, and the payment of the annuity in the event of survivance, are the only contingencies really provided for by the formula above deduced; or, in fact, that the ordinary method of valuation would have conducted Mr. Stephenson to precisely the same result as he has arrived at by his mode of solution.

The value of the deferred annuity alone is $\frac{N_{x+n}}{D_x}$, and the value of $P_x(1+i)$, payable in the event of death, is $P_x(1+i)\frac{M_x - M_{x+n}}{D_x}$. Therefore,

$$P_x = \frac{N_{x+n}}{D_x} + P_x(1+i)\frac{M_x - M_{x+n}}{D_x};$$

whence
$$P_x\left\{1 - \frac{(M_x - M_{x+n})(1+i)}{D_x}\right\} = \frac{N_{x+n}}{D_x},$$

and
$$P_x = \frac{N_{x+n}}{D_x - (M_x - M_{x+n})(1+i)}.$$

By substituting for $M_x(1+i)$ and $M_{x+n}(1+i)$ their equivalents $D_x - N_x i$ and $D_{x+n} - N_{x+n} i$, we have

$$P_x = \frac{N_{x+n}}{(N_x - N_{x+n})i + D_{x+n}},$$

which is the formula previously obtained.

This proves conclusively that the option of withdrawal does not enter in any way in the calculation of the premium; and indeed a little reflection will show that it cannot—for the sum which the policyholder is entitled to receive in the event of withdrawal does not admit of being fixed arbitrarily (as Mr. S. assumes), but can necessarily be no other than the surrender value of the policy (whatever it may be) determined according to the usual methods of calculation. No wonder then that (as Mr. Stephenson naively remarks "no method of deducing premiums returnable *at the option*, as well as on the death of a purchaser, has hitherto been published in any work on life annuities;" nor, it must be added, has Mr. Stephenson yet succeeded in supplying the omission.

It is true that, under the supposed arrangement between A and B, the former would have the option of withdrawing his deposit from B's hands at any time before the expiration of n years, and he would be entitled in

addition to an allowance from the Office for the surrender of the deferred annuity secured by the annual interest. This, however, merely shows, that in assurances of this description the value of the policy always exceeds the premium paid upon it—a circumstance which does not depend upon the mode of computing the premium, but arises from the nature of the contingency itself.

As it is P_x and not $P_x(1+i)$ that the representatives of A are to receive in the event of his death, the proper formula for the proposed benefit will be

$$P_x = \frac{N_{x+n}}{D_x - (M_x - M_{x+n})} = \frac{N_{x+n}}{(N_{x-1} - N_{x+n-1})(1-v) + D_{x+n}},$$

This formula may also be deduced by Mr. Stephenson's method, by supposing B to pay the interest at the beginning instead of the end of the year; the annual interest per £1 being in this case $\frac{i}{1+i}$, or $1-v$, instead of i.

Although Mr. Stephenson's claim to a solution of a new and impossible problem cannot be allowed, yet I think he is fairly entitled to the credit of having treated an old and perfectly practicable one in an original and striking manner.

I am, Sir,

Your very obedient servant,

London, 10th May, 1865. W. M. MAKEHAM.

THE D, N, &c., COLUMNS OF THE EQUITABLE EXPERIENCE.

(TABLE A, INTEREST 3 PER CENT.)

To the Editor of the Assurance Magazine.

SIR,—In looking over some of the early Numbers of the *Assurance Magazine*, I have found some tables in volume iii., page 366, constructed by the late Mr. Peter Hardy from the table of mortality known as the Equitable Experience; and as, in introducing these, you observe that space will be afforded to those contributors who may have authentic and original tables to offer, I am induced to send you the enclosed, in case you may consider any of them worthy of insertion.

The D, N, &c., columns have not, that I know of, appeared in print before.

The tables of annuities and assurance premiums will be found to vary, between the ages of about 85 to 93, from those of Mr. Hardy, who has not tabulated all the values between those ages quite correctly.

I am, Sir,

Your obedient servant,

London. W. MORGAN.

Preparatory Table for finding the Value of Annuities, &c., according to the Equitable Experience. (Table A, 3 per Cent.)

Age.	D.	N.	S.	M.	R.
10	3720·470	86696·263	1741952·276	1086·97296	37046·758017
11	3586·100	83110·163	1655256·013	1060·96579	35959·785059
12	3456·401	79653·762	1572145·850	1035·71611	34898·819271
13	3331·214	76322·548	1492492·088	1011·20186	33863·103163
14	3210·389	73112·159	1416169·540	987·40162	32851·901305
15	3093·775	70018·384	1343057·381	964·29459	31864·499687
16	2981·231	67037·153	1273038·997	941·86057	30900·205099
17	2872·618	64164·535	1206001·844	920·07998	29958·344531
18	2767·804	61396·731	1141837·309	898·93377	29038·264553
19	2666·657	58730·074	1080440·578	878·40347	28139·330785
20	2569·610	56160·464	1021710·504	859·02482	27260·927317
21	2476·490	53683·974	965550·039	840·74814	26401·902499
22	2387·137	51296·837	911866·065	823·52567	25561·154361
23	2300·888	48995·949	860569·228	806·80486	24737·628673
24	2217·638	46778·311	811573·279	790·57105	23930·823815
25	2137·285	44641·026	764794·968	774·81007	23140·252767
26	2059·269	42581·757	720153·942	759·04445	22365·442699
27	1983·984	40597·773	677572·185	743·73802	21606·398251
28	1911·337	38686·436	636974·412	728·87741	20862·660233
29	1841·239	36845·197	598287·976	714·44963	20133·782825
30	1773·603	35071·594	561442·779	700·44208	19419·333197
31	1707·945	33363·649	526371·185	686·44253	18718·891119
32	1644·608	31719·041	493007·536	672·85073	18032·448591
33	1583·133	30135·908	461288·494	659·27778	17359·597863
34	1523·479	28612·429	431152·586	645·73412	16700·320085
35	1465·601	27146·828	402540·157	632·22955	16054·585967
36	1409·803	25737·025	375393·329	619·11831	15422·356419
37	1355·676	24381·349	349656·304	606·05398	14803·238111
38	1303·181	23078·168	325274·955	593·04493	14197·184133
39	1251·963	21826·205	302196·787	579·78328	13604·139205
40	1202·316	20623·889	280370·582	566·60133	13024·355927
41	1154·499	19469·390	259746·693	553·80333	12457·754599
42	1108·159	18361·231	240277·303	541·08912	11903·951271
43	1063·539	17297·692	221916·072	528·74523	11362·862153
44	1020·577	16277·115	204618·380	516·76087	10834·116925
45	978·9519	15298·163	188341·264	504·86113	10317·356057
46	938·3722	14359·791	173043·101	492·79451	9812·494929
47	899·3256	13460·465	158683·310	481·07935	9319·700421
48	861·5158	12598·949	145222·845	469·46341	8838·621073
49	824·9104	11774·039	132623·895	457·95085	8369·157665
50	789·4787	10984·560	120849·856	446·54550	7911·206817
51	754·9682	10229·592	109865·296	435·02942	7464·661319
52	721·1532	9508·4391	99635·7035	423·20372	7029·631901
53	688·0410	8820·3981	90127·2644	411·09620	6606·428183
54	655·4355	8164·9626	81306·8663	398·53065	6195·331985
55	623·7520	7541·2106	73141·9037	385·93755	5796·801337
56	592·9761	6948·2345	65600·6931	373·32917	5410·863789
57	562·7218	6385·5127	58652·4586	360·34614	5037·534621
58	532·8267	5852·6860	52266·9460	346·84090	4677·188483
59	503·4963	5349·1897	46414·2600	333·02972	4330·347585
60	474·5739	4874·6158	41065·0703	318·77214	3997·317867
61	446·2498	4428·3660	36190·4545	304·27067	3678·545729
62	418·8531	4009·5129	31762·0885	289·87159	3374·275061
63	392·5185	3616·9944	27752·5757	275·73657	3084·403473
64	367·0610	3249·9334	24135·5813	261·71164	2808·666905
65	342·4606	2907·4728	20885·6479	247·80238	2546·955267
66	318·2711	2589·2017	17978·1751	233·58750	2299·152889
67	294·5103	2294·6914	15388·9734	219·09660	2065·565391

Preparatory Table for finding the Value of Annuities, &c. (continued).

Age.	D.	N.	S.	M.	R.
68	271·4615	2023·2299	13094·2820	204·62580	1846·468793
69	249·1152	1774·1146	11071·0522	190·18622	1641·842995
70	227·3352	1546·7795	9296·93748	175·66202	1451·656777
71	206·6127	1340·1668	7750·15800	161·56086	1275·994759
72	186·9044	1153·2624	6409·99122	147·87041	1114·433901
73	168·1689	985·0935	5256·72884	134·57871	966·563493
74	150·3662	834·72728	4271·63535	121·67415	831·984785
75	133·4579	701·26938	3436·90807	109·14545	710·310637
76	117·5128	583·75658	2735·63869	97·08743	601·165189
77	102·8967	480·85988	2151·88211	85·89408	504·077761
78	89·43116	391·42872	1671·02222	75·425548	418·183683
79	77·04997	314·37875	1279·59350	65·649109	342·758135
80	65·78398	248·59477	965·21475	56·627307	277·109026
81	55·38262	193·21215	716·61998	48·141997	220·481719
82	45·79711	147·41504	523·40782	40·169578	172·339722
83	37·15302	110·26202	375·99278	32·859378	132·170144
84	29·14061	81·12141	265·73076	25·929089	99·310766
85	22·37407	58·74734	184·60935	20·011309	73·381617
86	16·92143	41·82591	125·86201	15·210346	53·370368
87	12·60798	29·21793	84·036094	11·389746	38·16002265
88	9·124926	20·093006	54·818162	8·273917	26·77027700
89	6·410280	13·682726	34·725155	5·825047	18·49636035
90	4·685162	8·997564	21·042429	4·286636	12·67131370
91	3·326662	5·670902	12·044865	3·064597	8·38467805
92	2·306977	3·363925	6·3739623	2·141806	5·32008140
93	1·535852	1·828073	3·0100369	1·43787345	3·17827575
94	·9940790	·8339944	1·1819635	·94083405	1·74040230
95	·5328830	·2911114	·3479691	·51859185	·79956825
96	·2342537	·0568577	·0568577	·22577475	·28097640
97	·0568577	·0000000	·0000000	·05520165	·05520165

Table showing the Value of an Annuity upon a Single Life according to the Equitable Experience. (*Table A*, 3 *per Cent.*)

Age.	Annuity.	Age.	Annuity.	Age.	Annuity.	Age.	Annuity.
10	23·3025	32	19·2867	54	12·4573	76	4·9676
11	23·1756	33	19·0356	55	12·0901	77	4·6732
12	23·0453	34	18·7810	56	11·7176	78	4·3769
13	22·9113	35	18·5226	57	11·3475	79	4·0802
14	22·7736	36	18·2558	58	10·9842	80	3·7790
15	22·6320	37	17·9846	59	10·6241	81	3·4887
16	22·4864	38	17·7091	60	10·2716	82	3·2189
17	22·3366	39	17·4336	61	9·9235	83	2·9678
18	22·1825	40	17·1535	62	9·5726	84	2·7838
19	22·0238	41	16·8639	63	9·2148	85	2·6257
20	21·8556	42	16·5691	64	8·8539	86	2·4718
21	21·6774	43	16·2643	65	8·4899	87	2·3174
22	21·4888	44	15·9489	66	8·1352	88	2·2020
23	21·2944	45	15·6271	67	7·7915	89	2·1345
24	21·0938	46	15·3029	68	7·4531	90	1·9204
25	20·8868	47	14·9673	69	7·1217	91	1·7047
26	20·6781	48	14·6242	70	6·8040	92	1·4581
27	20·4628	49	14·2731	71	6·4864	93	1·1902
28	20·2405	50	13·9137	72	6·1703	94	·83896
29	20·0111	51	13·5497	73	5·8578	95	·53623
30	19·7742	52	13·1850	74	5·5513	96	·24272
31	19·5344	53	12·8196	75	5·2546		

Single Premiums for the Assurance of £1 upon a Single Life according to the Equitable Experience. (Table A, 3 per Cent.)

Age.	Single Premium.	Age.	Single Premium.	Age.	Single Premium.	Age.	Single Premium.
10	·29216	32	·40912	54	·60804	76	·82619
11	·29585	33	·41644	55	·61874	77	·83476
12	·29965	34	·42385	56	·62958	78	·84339
13	·30355	35	·43138	57	·64036	79	·85203
14	·30756	36	·43915	58	·65094	80	·86081
15	·31169	37	·44705	59	·66143	81	·86926
16	·31593	38	·45507	60	·67170	82	·87712
17	·32029	39	·46310	61	·68184	83	·88443
18	·32478	40	·47126	62	·69206	84	·88979
19	·32940	41	·47969	63	·70248	85	·89440
20	·33430	42	·48828	64	·71299	86	·89888
21	·33949	43	·49716	65	·72359	87	·90338
22	·34498	44	·50634	66	·73393	88	·90674
23	·35065	45	·51572	67	·74394	89	·90870
24	·35649	46	·52516	68	·75379	90	·91494
25	·36252	47	·53493	69	·76345	91	·92122
26	·36860	48	·54493	70	·77270	92	·92840
27	·37487	49	·55515	71	·78195	93	·93621
28	·38134	50	·56562	72	·79116	94	·94644
29	·38803	51	·57622	73	·80026	95	·95526
30	·39493	52	·58684	74	·80919	96	·96380
31	·40191	53	·59749	75	·81783	97	·97087

Annual Premiums for the Assurance of £1 upon a Single Life according to the Equitable Experience. (Table A, 3 per Cent.)

Age.	Annual Premium.	Age.	Annual Premium.	Age.	Annual Premium.	Age.	Annual Premium.
10	·012022	32	·020167	54	·045183	76	·138445
11	·012238	33	·020785	55	·047268	77	·147140
12	·012462	34	·021427	56	·049505	78	·156856
13	·012695	35	·022096	57	·051862	79	·167717
14	·012937	36	·022806	58	·054317	80	·180125
15	·013189	37	·023548	59	·056902	81	·193656
16	·013452	38	·024324	60	·059593	82	·207904
17	·013725	39	·025123	61	·062419	83	·222904
18	·014010	40	·025960	62	·065458	84	·235159
19	·014307	41	·026853	63	·068771	85	·246684
20	·014627	42	·027792	64	·072356	86	·258911
21	·014970	43	·028797	65	·076248	87	·272313
22	·015340	44	·029875	66	·080340	88	·283179
23	·015728	45	·031017	67	·084619	89	·289904
24	·016135	46	·032213	68	·089174	90	·313288
25	·016563	47	·033502	69	·094001	91	·340603
26	·017003	48	·034877	70	·099014	92	·377683
27	·017466	49	·036348	71	·104450	93	·427439
28	·017954	50	·037926	72	·110337	94	·514659
29	·018468	51	·039604	73	·116694	95	·621817
30	·019010	52	·041371	74	·123515	96	·775561
31	·019573	53	·043235	75	·130756	97	·970873

INSTITUTE OF ACTUARIES.

PROCEEDINGS OF THE INSTITUTE.

First Ordinary Meeting. Session 1864–5.—*Monday, 28th November,* 1864.

The President in the Chair.

Read and confirmed the minutes of the last anniversary meeting.

Mr. Wm. F. Purdy, nominated at the last ordinary meeting, was elected an Associate.

Mr. Peter Gray read a paper by Mr. Makeham "On the solution of general problems in survivorships."

Thanks having been voted to Mr. Gray and Mr. Makeham, the meeting adjourned to Tuesday, the 27th December, 1864.

Second Ordinary Meeting. Session 1864–5.—*Tuesday, 27th December,* 1864.

W. B. Hodge, Esq., Vice-President, in the Chair.

Read and confirmed the minutes of last ordinary meeting.

The following gentlemen, nominated at the last ordinary meeting, were elected members of the Institute, viz.:—

Fellows.

Henry Ambrose Smith. | Augustus Hendriks.
Edward W. Brabrook.

Associates.

Howard S. Smith.	George W. Cooke.
Edward H. Panton.	Alfred B. Adlard.
Alexander Stuart.	Charles B. Coghlan.
John Hy. Bignold, M.A.	John B. Bannerman.

Mr. Peter Gray read a paper "On a table for the formation of logarithms and antilogarithms. Part. I."

Thanks were voted to Mr. Gray, and the meeting adjourned to Monday, the 30th January, 1865.

Third Ordinary Meeting. Session 1864–5.—*Monday, 30th January,* 1865.

The President in the Chair.

Read and confirmed the minutes of last ordinary meeting.

The following gentlemen, duly nominated at the last ordinary meeting, were elected Associates of the Institute, viz.:—

Alfred M. Humphreys.	Wm. Vaughan.
Edward H. Hoddinott.	Wm. Rodgers.
John Duncan.	Stephen H. Emmens.

The President announced the following as the result of the Examinations held in December, 1864, viz.:—

Matriculation Examination.

Ten candidates presented themselves for this Examination, of whom four passed in the following order of merit, viz.:—

	Marks.		Marks.
Wm. Vaughan	472	Geo. S. Crisford	266
H. S. Smith	295	Edward Smyth	250

Second Year's Examination.

Seven gentlemen presented themselves for this Examination, and of these four passed, viz.:—

	Marks.		Marks.
Charles F. Haycraft	575	Edward F. Day	465
Chas. J. Wilkins	535	F. Addiscott	395

THIRD YEAR'S EXAMINATION.

One candidate presented himself, and passed, viz., Mr. C. G. Laing.

Mr. Peter Gray read the second part of his paper "On a table for the formation of logarithms and antilogarithms."

Thanks having been voted to Mr. Gray, the meeting adjourned to 27th February, 1865.

Fourth Ordinary Meeting. Session 1864–5.—*Monday, 27th February,* 1865.

The President in the Chair.

Read and confirmed the minutes of last ordinary meeting.

Mr. W. S. B. Woolhouse read the third part of his paper "On interpolation, summation, and the adjustment of numerical tables."

Thanks were voted to Mr. Woolhouse, and the meeting adjourned to the 27th March, 1865.

Fifth Ordinary Meeting. Session 1864–5.—*Monday, 27th March,* 1865.

The President in the Chair.

Read and confirmed the minutes of last ordinary meeting.

The following gentlemen, nominated at the last ordinary meeting, were elected Associates of the Institute, viz.:—

Alfred Sprulls. | Oscar L. Wood.

Mr. Archibald Day read a paper "On the statistics of second marriages in the families of the peerage."

Thanks were voted to Mr. Day and the meeting adjourned to Monday, the 24th April, 1865.

Sixth Ordinary Meeting. Session 1864–5.—*Monday, 24th April,* 1865.

The President, in the Chair.

Read and confirmed the minutes of last ordinary meeting.

The following gentlemen, nominated at the last ordinary meeting, were elected Associates of the Institute, viz.:—

Geo. Hamilton Bell. | Geo. S. Horsnail.

Mr. Berridge read a paper "On the graduation of the table of mortality deduced by Mr. Bailey and Mr. Day from observations amongst the families of the peerage"; and Mr. Adler read one "On the Government insurance rates and regulations."

Thanks were voted to both gentlemen, and the meeting adjourned to Monday, the 27th November, 1865.

The Eighteenth Annual General Meeting, Saturday, 3rd June, 1865.

ARTHUR HUTCHESON BAILEY, Esq., in the Chair.

Mr. J. Hill Williams, Honorary Secretary, read the minutes of the preceding ordinary meeting, and the following Reports and accounts of Receipts and Payments:—

"The Council are glad to be enabled to report, that the affairs of the Institute continue to be in a very satisfactory state.

"For particulars, they refer to the letter from the Assistant-Secretary, which accompanies this report. In that document it will be seen, amongst other matters, that the Society's income for the year was £640, which, however, includes £150. 13*s.* received on account of the Hardy Memorial, and that the expenses were £460, being greater than usual in consequence of the cost of the Index to the *Journal*. The property of the Institute is now estimated at £1,110; and the number of members on the register is 203, being an increase of 5 in the twelve months.

"The following papers have been read during the Session, viz.:—

'On the solution of general problems in survivorships.' By Mr. W. M. Makeham.

'On a table for the formation of logarithms and anti-logarithms.' Parts I. and II. By Mr. Peter Gray.

'On interpolation, summation, and the adjustment of numerical tables.' By Mr. W. S. B. Woolhouse.

'On the statistics of second marriages amongst the families of the peerage.' By Mr. Archibald Day.

'On a method of graduation applied to the peerage mortality deduced by Mr. Bailey and Mr. Day, with tables.' By Mr. Wm. Berridge.

'On the Government assurance tables.' By Mr. Adler.

"Since the last annual meeting an Index to the first ten volumes of the *Journal* has been completed, and appears to have given general satisfaction. The Council has also received information that the Clerical and Medical Office, the London Assurance, the Pelican, and the Guardian Offices, have all, at the cost of no inconsiderable labour and expense, completed the extracts from their registers of the particulars required for determining the rate of mortality amongst the lives assured; and the attention of the Council is now being directed to the means of using, to the greatest advantage, the valuable data thus placed at their disposal. Meanwhile they are in hopes of receiving further contributions of the like nature from other Assurance Companies.

"As above observed, the contributions to the Hardy Testimonial have reached the sum of £150. 13*s*., and it is now proposed to close the subscription list and to proceed to carry out the objects for which the fund has been raised."

"*Institute of Actuaries*, 12, *St. James's Square*,
2nd May, 1865.

"Dear Sir,—Herewith I transmit to you the Abstract of Receipts and Payments of the Institute of Actuaries for the financial year ended on the 31st March, 1865 (see p. 244), duly audited, with a few remarks thereon.

"Receipts.

"In 1862-63 the receipts were £426. 8*s*., in 1863-64 they were £521. 6*s*. 6*d*., and in 1864-65 they were £640. 1*s*. 11*d*.; after deducting from this last amount £150. 13*s*., which was received for the Hardy Memorial Fund, there appears a slight decrease to the previous year. This is accounted for by the number of life subscriptions which were received in 1863-64.

"The total sums received from subscriptions in the last three years are as follows:—

1862–63.	Annual subscriptions		£409 10 0
1863–64.	Annual subscriptions . .	£411 12 0	
	Life " . .	84 0 0	
			£495 12 0
1864–65.	Annual subscriptions . .	£415 16 0	
	Life " . .	21 0 0	
			£436 16 0

"The balance in the bank on the 31st of March, 1864, amounted to £159. 12*s*. 8*d*., as compared with £417. 5*s*. 11*d*. of the previous year—a decrease of £217. 13*s*. 3*d*. This falling-off is caused by the large sum of £488. 18*s*. 2*d*. having been invested during the year.

"Payments.

"Had it not been for the cost of having the General Index made and printed (£96. 8*s*. 6*d*.), the annual subscriptions alone would have covered the general expenses. The following table shows the annual subscriptions and general expenses for the last three years:—

	Annual Subscriptions.	General Expenses.	Difference.
1862–63........	£409 10 0	£371 18 0	+ £37 12 0
1863–64........	411 12 0	399 3 6	+ 22 8 6
1864–65........	415 16 0	460 2 11	− 44 6 11

"The cost of printing the *Journal* was £85. 3*s*. 6*d*., as compared with £104. 9*s*. of the previous year. The payments for salaries, lighting, postage stamps, and miscellaneous expenses, are also less; whilst for stationery and sundry printing, ordinary meeting expenses, and the library, they have slightly increased.

"ASSETS.

"The investments of the Institute on the 31st March last amounted to £646. 14*s*. 3*d*., as compared with £203. 17*s*. 8*d*. of the previous year; and the balance at the bankers, and the petty cash balance, amounted to £159. 15*s*. 10*d*., as compared with £422. 13*s*. 5*d*. The total available assets, therefore, at the end of the year amounted to £806. 10*s*. 1*d*., against £627. 11*s*. 1*d*. for the year 1863-64.

"LIABILITIES.

"The following is an estimate of the liabilities of the Institute to the 31st September next:—

Two quarters' salaries	£52	10	0
Rent (half year, due Michaelmas)	37	10	0
Journals, April and July (say)	45	0	0
Stationery and sundry printing (say)	10	0	0
Miscellaneous (say)	10	0	0
	£155	0	0

"LIST OF MEMBERS.

"On the 1st April, 1864, the total number of Members on the list was 198; during the year, 22 new members were elected—viz., 4 Fellows and 22 Associates; and the losses were 5 deaths and 12 resignations, of whom 4 were Fellows and 13 Associates; leaving on the list, on the 31st March, 1865, 90 Fellows and 113 Associates—in all, 203 members. This is an increase of 5 in twelve months.

"In the course of the year 13 Associates have been transferred to the Fellows. These transfers, although they affect the totals of the different classes of members, do not alter the general total.

"I am, dear Sir, yours faithfully,

"FREDERICK GOVER,

"C. JELLICOE, Esq." "*Assistant Secretary.*

The Chairman moved, "That the Report of the Council and the Abstract of Receipts and Payments be adopted, entered on the minutes, and printed in the *Journal*."

Mr. Hodge seconded the motion, and it was at once put and carried unanimously.

The Chairman said the next business was to elect the President, Vice-Presidents, Council, and officers for the ensuing year. The Council had put forward a list, but the members would understand that this was merely a recommendation in accordance with the usual custom, and they were quite at liberty to make any alteration in it they thought proper.

Mr. R. P. Hardy and Mr. B. Newbatt having been appointed scrutineers, a ballot was taken, and the result thereof showed that the whole of the gentlemen recommended by the Council were elected unanimously.

Mr. Galsworthy said he had been requested to move a resolution, and in doing so he should, if not deemed out of order, avail himself of the opportunity of making one or two remarks upon the Report which had just been passed. Looking at the way in which the Chairman had put the motion for the adoption of the Report, he was disposed to think that, according to city notions, he was one of the most admirable chairmen they could have obtained; for in the city it was thought to be an excellent plan to put the resolution before anyone had an opportunity of making remarks upon it, lest those remarks should be of an unfavourable nature. (Laughter.) As, however, his observations would be of a satisfactory character, it made no difference whether he offered them then or before the Report was adopted. It seemed hardly right that the Report should be passed over quite *sub silentio*. (Hear, hear.) So far as he had been able to gather, the Report was really a very admirable one. There had been an increase in the receipts; and although, after deducting the amount for the Hardy Memorial, that increase was not very large, yet it was satisfactory to find that there was an increase, and that the expenditure was considerably under the income. (Hear, hear.) If they looked to the papers that had been read in this Institute during the past Session they had equal cause for congratulation. Mr. Woolhouse had favoured them with a paper, which, for complete-

ness and ingenuity, was seldom approached; and he had illustrated his views by elaborate diagrams that must have cost him an immense amount of labour. (Hear, hear.) Then they had been favoured with two excellent papers by Mr. Peter Gray, another by Mr. A. Day, and other gentlemen had favoured them with their productions. So that whether they looked at the financial position of this Institute, or at the quality of the papers read during the past Session, they had much cause for congratulation. (Hear, hear.) With regard to the mortality experience of the Offices, it was satisfactory to find that such large Companies as the Clerical, Medical and General, the Guardian, the London Assurance Corporation, and the Pelican, had been amongst the first to supply the facts that were so desirable to obtain in a combined form. It was very necessary that the data should be added to as much as possible, and perhaps ultimately the experience of other Offices might be obtained. (Hear, hear.) He was very glad the number of members increased. It was not every Institution that could keep up its lapses in these days, and whose new business was improved upon year by year. He could not sit down without expressing his deep regret that the President was not present on this occasion. (Hear, hear.) They were always much pleased with the observations he made on the proceedings and prospects of the Institute, and he was sure no one could listen to what he said without feeling gratified that the Institute enjoyed so able and judicious a President. (Cheers.) They were all greatly indebted to him for the great, the extraordinary amount of attention which he gave at all times and under all circumstances, in the best possible manner, to promote the interests of the Institute of Actuaries. (Cheers.) It was, therefore, with great pleasure that he (Mr. Galsworthy) proposed, "That the best thanks of this meeting be given to the retiring President, Vice-Presidents, Council, and other officers, for their services during the past year."

Mr. Newbatt seconded the motion, and in doing so alluded to the loss the Institute was about to experience by the resignation of Mr. Reddish. He thought they ought not to take leave of their Honorary Secretary without some acknowledgment of the services he had rendered. (Hear, hear.) They would all feel that he had honourably earned the retirement which he was about to seek, and that his connexion with this Institute for so many years had been of a most useful and satisfactory character. Mr. Galsworthy had referred to the peculiar excellence of the papers that had been read before them during the past Session; and he must say that, as one of the objects of this Institute was to raise the status of the members of the actuarial profession, he did not know any means whereby that could be more satisfactorily accomplished than by having papers presented to them of the character of those with which they have been favoured during the Session which had lately terminated. (Hear, hear.) Those papers would have a tendency to extend a knowledge of the Institute far beyond its walls, for most of them were of a character to interest the scientific world at large. (Hear, hear.)

The resolution having been carried unanimously—

The Chairman said that, in the absence of the President, it had devolved somewhat unworthily upon him to request them, on behalf of the Council and officers, to accept their thanks for the vote just passed. He was very glad Mr. Galsworthy had found an opportunity to offer the remarks he had. The three points he had alluded to—the state of their finances, the character of the papers, and the data they were engaged in obtaining—showed that they might be fairly considered to be making progress. (Hear, hear.) They could not boast of being rich, and it was not desirable they should be. The accumulation of money was not their object. (Hear, hear.) So long as they could make both ends meet that was all they wished. He was sure the papers read during the past Session were a credit not only to this Institute but to any scientific Society.' (Hear, hear.) With regard to the mortality experience which was at present being obtained, there had been a great deal of discussion, and a great deal more would be required before they had completed their investigations. He hoped that in the main the members would be satisfied with the progress that had been made not only in this matter, but in the Institute generally. He thought this Institute formed a centre for gentlemen connected with all classes of assurance business, old and young, home and foreign, London and provincial. (Hear, hear.)

On the motion of Mr. Hodge, seconded by Mr. Brown, Mr. G. W. Berridge, Mr. R. P. Hardy, and Mr. B. Newbatt, were elected auditors for the ensuing year.

Mr. Hodge next proposed, "That the best thanks of the members are due and are hereby given to Mr. John Reddish, for his long and valued services as Honorary Secretary of this Institute."

Mr. Lodge seconded the resolution, which was carried unanimously.

Mr. Reddish briefly returned thanks, and the proceeding closed with a vote of thanks to the Chairman.

INSTITUTE OF ACTUARIES.

Dr. *Abstract of Receipts and Payments for the Year ended 31st March, 1865.* **Cr.**

RECEIPTS.

	£	s.	d.	£	s.	d.
To Cash balance brought forward	417	5	11			
Petty cash „ „	5	7	6			
				422	13	5
To Subscriptions for 1863-64 (arrears)				22	1	0
„ Subscriptions for 1864-65, viz.:—						
61 Fellows, Town, at £3 3 0	192	3	0			
23 „ Country, „ 2 2 0	48	6	0			
74 Associates, Town, „ 2 2 0	155	8	0			
19 „ Country, „ 1 1 0	19	19	0			
				415	16	0
Paid into Bank in error				1	1	0
Composition—T. J. Searles (Associate, Town)				21	0	0
Dividends on Consols				29	10	11
Hardy Memorial Fund				150	13	0
				£1,062	15	4

PAYMENTS.

	£	s.	d.	£	s.	d.	£	s.	d.
By Investment in Consols							442	16	7
Rent				75	0	0			
Salaries				95	5	0			
Journal—Printing	£85	3	6						
Index „	75	8	6						
„ compiling	21	0	0						
				181	12	0			
Stationery and sundry Printing				32	11	3			
Ordinary Meeting Expenses				19	5	11			
Lighting				3	14	8			
Postage and Receipt Stamps				6	10	0			
Library				8	8	7			
Advertising Examinations				4	18	0			
Coals				8	8	0			
Furniture and Repairs				10	14	6			
House and Petty Expenses				4	12	0			
Miscellaneous (including Check Book)				9	3	0			
							460	2	11
Balance in Bank, carried forward				159	12	8			
„ of Petty Cash				0	3	2			
							159	15	10
							£1,062	15	4

31st March, 1864. **ASSETS.**

	£	s.	d.
£700, 3 per Cent. Consols (cost)	646	14	3
Cash balance	159	15	10
Books in Library (say)	303	9	11
	£1,110	0	0

Examined and found correct:—

(Signed) WILLIAM KING,
(„) JAMES TERRY,
(„) JOHN COLES,
Auditors.

THE

ASSURANCE MAGAZINE,

AND

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the Summation of Divergent Series. By PROFESSOR DE MORGAN.

IN the last Number I gave the most elementary view I could arrive at of Arbogast's method of development. In the communication following I saw that Mr. Peter Gray had referred to Stirling's theorem; and this suggested that it might be useful to give, by means of common algebra only, an account of the two most important cases of summation of many terms of a divergent series.

The method I use depends upon a theorem which, simple as it is, I cannot find mentioned by any writer. Stated in the language of infinites, it would be—The infinite sums of two diverging series are in the ratio of their last terms. Stated in the language of limits, it is—If $a_1+a_2+a_3+\ldots$ and $b_1+b_2+b_3+\ldots$ be diverging series, then the ratio of $a_1+\ldots+a_n$ to $b_1+\ldots+b_n$, as n increases without limit, continually approaches to the value of a_n to b_n. And this, be it understood, whether the diverging series have their terms increasing or diminishing.

Some persons, even though well-informed mathematicians, confess to having but a cloudy idea of terms which diminish, and diminish without limit, mounting up to any sum we please. The difficulty arises from not looking at both sides of the question: we may get *as much as we please;* but we are at liberty to take *as*

many terms as we want. It is the old story of the pendulum which revolted in disgust at the number of times it would have to tick before the end of the year, and was brought to reason by being reminded that it would have just as many seconds to tick in.

The series $1+\frac{1}{2}+\frac{1}{3}+\ldots$ is of this kind. Parcel it out as follows:—

$$1+\tfrac{1}{2}+(\tfrac{1}{3}+\tfrac{1}{4})+(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8})+(\tfrac{1}{9}+\ldots+\tfrac{1}{16})+\ldots$$

Each parcel is obviously more than half a unit. If then we want to exceed a million of millions, we have nothing to do but to sum two millions of millions of these lots. I mention this common proof that I may give another which I never found in a book, though it is not mine. It is well known that when $a-b+c-d+\ldots$ consists of terms diminishing without limit, the series is convergent, with a limit between a and $a-b$. Now

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots \text{ is } \begin{array}{l} 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots \\ \quad\; +1 \quad\; +\frac{1}{2}+\ldots \end{array}$$

And if it be S, we have $S=a+S$, where a is finite. Hence S is infinite.* An objection may be made of a kind which would have counted for nothing in the last century—and which would be met by saying we have shown that the series cannot be convergent, for then its infinitely small deferred remainder would be a finite quantity.

Carry the terms only as far as $(2n)^{-1}$, and the preceding method, since $1-\frac{1}{2}+\frac{1}{3}-\ldots$ is $\log 2$,† gives the following:—The greater n is, the more nearly

$$\frac{1}{n+1}+\ldots\ldots+\frac{1}{2n}=\log 2.$$

This is just what would be true if the equation $1+\ldots+n^{-1}=\log n$ were nearly true when n is great: and we have here the easiest presumption of the connection between $1+\ldots+n^{-1}$ and $\log n$.

* This ingenious proof was given me, 37 years ago, by a pupil of the age of 13, whose mathematical power was singularly in advance of his years. Of many things as worthy of remark in one so young, I only remember what is here given. Time and thought have developed this boy into Professor Sylvester, whose inventive power, in everything to which his taste has led him, places him in the highest rank.

The divergence of the series was first noticed and proved by John Bernoulli, and another proof was given by James Bernoulli. Both are much more difficult than that by collection into lots each greater than half a unit. I do not know who first gave this.

† In this *Journal*, supported by contributors who have constantly to think of the common logarithm, it is very common to distinguish the Naperean logarithm when it is used. But it is now so well established, in algebraical writing, that $\log x$ shall mean the Naperean logarithm, that it would be a good thing if, without further mention, the common logarithm were always denoted by c. $\log x$.

I now proceed to the theorem stated. Since the series are divergent, if we begin at a_m and b_m instead of a_1 and b_1, m being any given number, however great, $a_m+a_{m+1}+\ldots$ and $b_m+b_{m+1}+\ldots$, the same number of terms being taken out of both, have the same limiting ratio as

$$(a_1+\ldots+a_{m-1})+a_m+\ldots \text{ and } (b_1+\ldots+b_{m-1})+b_m+\ldots$$

This because two quantities which increase without limit have a limiting ratio which is not altered by adding any given quantities to both. Let l be the limit of $a_n : b_n$, and take m so great that at and after $n=m$, $a_n : b_n$ shall lie between $l\pm a$. Then, by a common theorem of arithmetic, $a_m : b_m$, $a_{m+1} : b_{m+1}$, &c., all lying between $l\pm a$, so does $a_m+a_{m+1}+\ldots$ divided by $b_m+b_{m+1}+\ldots$ This last fraction then, and consequently $(a_1+\ldots):(b_1+\ldots)$, has a limit between $l\pm a$, however small a may be; its limit is therefore l, the limit of $a_n : b_n$.

Next, let c_n be a quantity which increases without limit with n; or better, let $c_\infty=\infty$, a symbol which those who choose can translate into the language of limits. Then $c_1+(c_2-c_1)+(c_3-c_2)+\ldots$ is a divergent series; for the sum of n terms is c_n. That is, $a_1+\ldots+a_n$ divided by c_n has the same limit as a_n divided by c_n-c_{n-1}; or the equation

$$a_1+a_2+\ldots+a_n=\frac{c_n}{c_n-c_{n-1}}\,a_n$$

has sides which approach without limit to a ratio of equality as n increases without limit. If then we can contrive that a_n and c_n-c_{n-1} shall approach to a ratio of equality, we have a perpetual approach to truth in $a_1+\ldots+a_n=c_n$.

The process here divides as follows. Those who understand the integral calculus can be shown that c_n may be $\int a_n dn$: those who do not must be content to have the result of the integral calculus placed before them and verified.

Let $a_n=n^{-1}$; then c_n may be $\log n$: let $a_n=\log n$; then c_n may be $n\log n-n$. For in the first case

$$\frac{a_n}{c_n-c_{n-1}}=\frac{n^{-1}}{\log n-\log(n-1)}=\frac{n^{-1}}{-\log(1-n^{-1})}=\frac{n^{-1}}{n^{-1}+\frac{1}{2}n^{-2}+\ldots},$$

of which the limit is unity. In the second case

$$\frac{a_n}{c_n-c_{n-1}}=\frac{\log n}{n\log n-n-(n-1)\log(n-1)+n-1}.$$

Of this the denominator is

$$\log(n-1)-1+\{-n\log(1-n^{-1}) \text{ or } 1+\tfrac{1}{2}\frac{1}{n}+\ldots\},$$

and the limit of the fraction is unity.

Let us then assume, on trial,

$$1+\tfrac{1}{2}+\ldots+\frac{1}{n}=\log n+\mathrm{A}+\frac{\mathrm{B}}{n}+\frac{\mathrm{C}}{n^2}+\frac{\mathrm{D}}{n^3}+\ldots$$

Change n into $n+1$, and subtract, and we have

$$\frac{1}{n+1}=\log\left(1+\frac{1}{n}\right)+\mathrm{B}\left(\frac{1}{n+1}-\frac{1}{n}\right)+\mathrm{C}\left(\frac{1}{(n+1)^2}-\frac{1}{n^2}\right)+\mathrm{D}\left(\frac{1}{(n+1)^3}-\frac{1}{n^3}\right)+\ldots$$

$$\frac{1}{n}-\frac{1}{n^2}+\frac{1}{n^3}-\frac{1}{n^4}+\ldots=\frac{1}{n}-\tfrac{1}{2}\frac{1}{n^2}+\tfrac{1}{3}\frac{1}{n^3}-\tfrac{1}{4}\frac{1}{n^4}+\ldots$$

$$+\mathrm{B}\left(-\frac{1}{n^2}+\frac{1}{n^3}-\frac{1}{n^4}+\ldots\right)+\mathrm{C}\left(-\frac{2}{n^3}+\frac{3}{n^4}-\ldots\right)+\mathrm{D}\left(-\frac{3}{n^4}+\ldots\right)+\ldots$$

which is satisfied by

$$-1=-\tfrac{1}{2}-\mathrm{B},\quad 1=\tfrac{1}{3}+\mathrm{B}-2\mathrm{C},\quad -1=-\tfrac{1}{4}-\mathrm{B}+3\mathrm{C}-3\mathrm{D}, \text{ \&c.}$$
$$\mathrm{B}=\tfrac{1}{2},\quad \mathrm{C}=-\tfrac{1}{12},\quad \mathrm{D}=0, \text{ \&c.}$$

This method may be easily carried further, and the result is

$$1+\tfrac{1}{2}+\ldots+\frac{1}{n}=\log n+\mathrm{A}+\frac{1}{2n}-\tfrac{1}{6}\frac{1}{2n^2}+\tfrac{1}{30}\frac{1}{4n^4}$$
$$-\tfrac{1}{42}\frac{1}{6n^6}+\tfrac{1}{30}\frac{1}{8n^8}-\ldots$$

By the method employed, it is clear that if this equation be true for any one value of n, it is true for the preceding and following values; for we have so constructed the second side of the equation (call it ϕn) as to satisfy

$$\phi(n+1)=\phi n+\frac{1}{n+1}.$$

Take A so as to satisfy this equation when $n=10$; this gives

$$2{\cdot}9289683=\mathrm{A}+\log 10+\frac{1}{20}-\frac{1}{1200}+\frac{1}{1200000}-\ldots$$

which gives, $\log 10$ being $2{\cdot}3025851$, $\mathrm{A}={\cdot}5772157$. More accurately

$$\mathrm{A}={\cdot}57721{,}56649{,}01532{,}86060{,}65\ldots$$

This constant, which I usually denote by γ, has that sort of importance which attaches to what are known as π and ε; that is, $3{\cdot}14159\ldots$ and $2{\cdot}71828\ldots$.

Let us now take

$$\log 1+\log 2+\ldots+\log n=n\log n-n+\mathrm{A}+\frac{\mathrm{B}}{n}+\ldots$$

As before, change n into $n+1$, and subtract: we have then

$$\log(n+1)=(n+1)\log(n+1)-n-1-n\log n+n$$
$$+B\left(\frac{1}{n+1}-\frac{1}{n}\right)+C\left(\frac{1}{(n+1)^2}-\frac{1}{n^2}\right)+\ldots$$
$$0=n\log\left(1+\frac{1}{n}\right)-1+B\left(\frac{1}{n+1}-\frac{1}{n}\right)+\ldots$$

Develope as before, and we find that we cannot produce an identical equation: out of the first term comes $-\frac{1}{2}n^{-1}$, the only term having n^{-1}. But when we look at $n\log n-n$, we see that if, instead of $n\log n-n$, we write $n\log n-n+N\log n$, where N is any constant, we still have approach to equality between $\log 1+\ldots+\log n$ and $n\log n-n+N\log n$; and this is all that is demanded by our original theorem. Let us then assume

$$\log 1+\ldots+\log n=n\log n-n+N\log n+A+\frac{B}{n}+\ldots$$

Proceeding as before, we have

$$0=n\log\left(1+\frac{1}{n}\right)-1+N\log\left(1+\frac{1}{n}\right)+B\left(\frac{1}{n+1}-\frac{1}{n}\right)+\ldots$$

After development we see that this equation is identically satisfied if

$$-\tfrac{1}{2}+N=0,\ \tfrac{1}{3}-\tfrac{1}{2}N-B=0,\ -\tfrac{1}{4}+\tfrac{1}{3}N+B-2C=0,$$
$$\tfrac{1}{5}-\tfrac{1}{4}N-B+3C-3D=0;$$
$$\text{or } N=\tfrac{1}{2},\ B=\tfrac{1}{12},\ C=0,\ D=-\tfrac{1}{360},\ \&c.$$

By this, carried further, we have

$$\log 1+\ldots+\log n=n\log n-n+\tfrac{1}{2}\log n+A+\tfrac{1}{6}\cdot\frac{1}{2n}-\tfrac{1}{30}\frac{1}{3.4n^3}$$
$$+\tfrac{1}{42}\frac{1}{5.6n^5}-\ldots$$

If we were to take the trouble of determining A by assuming, as before, $n=10$, and calculation, we should find $A=\log\sqrt{(2\pi)}$. But this may be taken for granted until the reader becomes acquainted with Wallis's theorem, as follows:—The greater n is made, the more nearly is

$$\frac{\pi}{2}=\frac{4}{3}\cdot\frac{16}{15}\cdot\frac{36}{35}\cdot\ldots\ldots\cdot\frac{4n^2}{4n^2-1},$$
$$\text{or } \frac{2.4.6\ldots.2n}{1.3.5\ldots.2n-1}=\sqrt{(n+\tfrac{1}{2})\pi}.$$

Let n be taken exceedingly great, so that all which follows A may be rejected as exceedingly small: that is,

$$\log 1 + \ldots\ldots + \log n = n\log n - n + \tfrac{1}{2}\log n + \mathrm{A}$$

is as nearly true as we please. This gives

$$\begin{aligned}
1.2.3\ldots\ldots n &= n^{n+\frac{1}{2}}\varepsilon^{\mathrm{A}-n} \\
1.2.3\ldots\ldots 2n &= (2n)^{2n+\frac{1}{2}}\varepsilon^{\mathrm{A}-2n} \\
2.4.6\ldots\ldots 2n &= 2^n n^{n+\frac{1}{2}}\varepsilon^{\mathrm{A}-n} \\
1.3.5\ldots\ldots 2n-1 &= (2n)^n\sqrt{2}\,\varepsilon^{-n} \\
\frac{2.4.6\ldots\ldots 2n}{1.3.5\ldots\ldots 2n-1} &= \sqrt{\frac{n}{2}}\cdot\varepsilon^{\mathrm{A}} = \sqrt{(n+\tfrac{1}{2})\pi}
\end{aligned}$$

The greater n, the more nearly is this equation true: it becomes true, then, when n is made infinite; that is, $\varepsilon^{\mathrm{A}} = \sqrt{(2\pi)}$. And thus we have

$$1.2.3\ldots\ldots n = \sqrt{(2\pi)}.n^{n+\frac{1}{2}}\varepsilon^{-n}.\varepsilon^{\frac{1}{12n}-\frac{1}{360n^3}+\frac{1}{1260n^5}-\ldots\ldots}$$

This approaches to truth, if we stop at what is written, as n increases; but it is very near the truth at the beginning. If $n=1$, we have

$$1 = \sqrt{(2\pi)}.\varepsilon^{-1}.\varepsilon^{\cdot 0813492},\quad \sqrt{(2\pi)} = \varepsilon^{\cdot 9186508},$$

$$\text{or}\quad \pi = \tfrac{1}{2}\varepsilon^{1\cdot 8373016}.$$

This gives $\pi = 3\cdot 1398$ instead of $3\cdot 1416$.

There is a point about the demonstrations above which will not be quite clear at first to the young algebraist. We have assumed forms on one side of the equation: how do we know those forms are correct? As follows:—Let there be a series $\phi 1 + \phi 2 + \ldots + \phi n$. Suppose we find a function, ψn, which satisfies the equation $\psi(n+1) = \phi(n+1) + \psi n$. Let $\phi 1 + \ldots + \phi n$ be $\psi n + \chi n$. Then

$$\begin{aligned}
\phi 1 + \phi 2 + \ldots + \phi n + \phi(n+1) &= \psi(n+1) + \chi(n+1) \\
= \psi n + \chi n \quad + \phi(n+1) &= \psi(n+1) + \chi n.
\end{aligned}$$

Hence $\chi(n+1) = \chi n$; or χn is not changed by changing n into $n+1$. Step by step we find that if n be integer, χn is not changed by changing n into any other integer. Accordingly, since n is integer in every case we use, χn is, throughout our problem, a constant: that is, $\phi 1 + \ldots + \phi n$ differs from ψn by a constant. Now, in our demonstrations we assume a form for ψn which, by determination of constants, we find we can make to satisfy $\psi(n+1) - \psi n = \phi(n+1)$; and we have the constant A, which, not determined in the verification of this last equation, is subsequently determined.

The theorem of which the preceding contains particular cases is accessible to those who have an elementary knowledge of the integral calculus. Take the series $\phi 0 + \ldots + \phi n$: assume the form

$$\int\phi n\,dn + \mathrm{B}\phi n + \mathrm{C}\phi' n + \mathrm{D}\phi'' n + \mathrm{E}\phi''' n + \ldots\ldots$$

for ψn. Can we determine B, C, &c., so that $\psi(n+1)-\psi n = \phi(n+1)$? If so, by help of the arbitrary constant in $\int \phi n dn$ we can make good the equality of $\phi 1+\ldots+\phi n$ and ψn. Change n into $n+1$. Then, by Taylor's theorem, $\int \phi n dn$ becomes $\int \phi n dn + \phi n + \phi' n.\frac{1}{2}+\ldots$, ϕn becomes $\phi n+\phi' n+\phi'' n.\frac{1}{2}+\ldots$, $\phi' n$ becomes $\phi' n+\phi'' n+\phi''' n.\frac{1}{2}+\ldots$ It will be convenient to write C : 2, D : 2.3, E : 2.3.4, &c., for C, D, E, &c.; and also to write subscript numerals for fractional divisors: thus, M_4 may mean M : 2.3.4, 1_3 may mean 1 : 2.3. This being agreed on, in $\psi n = \int \phi n dn + B_1 \phi n + C_2 \phi' n + D_3 \phi'' n + E_4 \phi''' n + \ldots$ write $n+1$ for n, and subtract ψn from $\psi(n+1)$. We have

$$\psi(n+1)-\psi n = \phi n + (1_2+B_1)\phi' n + (1_3+1_2B_1+C_2)\phi'' n$$
$$+(1_4+1_3B_1+1_2C_2+D_3)\phi''' n + (1_5+1_4B_1+1_3C_2+1_2D_3+E_4)\phi^{\text{IV}} n+\ldots$$

This is $\phi(n+1)$ if the coefficients of ϕn, &c., be 1, 1_1, 1_2, 1_3, &c.; so that we have the following results:—

$$1_2+B_1=1_1,\quad B=\tfrac{1}{2},\quad 1_3+1_2B_1+C_2=1_2,\quad \frac{1}{2.3}+\tfrac{1}{2}.\tfrac{1}{2}+\frac{C}{2}=\tfrac{1}{2},$$
$$\text{or } C=\tfrac{1}{6}.$$
$$1_4+1_3B_1+1_2C_2+D_3=1_3 \text{ gives } D=0,$$
$$1_5+1_4B_1+1_3C_2+1_2D_3+E_4=1_4 \text{ gives } E=-\tfrac{1}{30}.$$

Similarly, it will be found that $F=0$, $G=\frac{1}{42}$, $H=0$, $I=-\frac{1}{30}$, $K=0$, $L=\frac{5}{66}$, &c. And we have

$$\phi 0+\ldots+\phi n = A+\int \phi n dn + \tfrac{1}{2}\phi n + \tfrac{1}{6}\frac{\phi' n}{2} - \tfrac{1}{30}\frac{\phi''' n}{2.3.4} + \tfrac{1}{42}\frac{\phi^{\text{V}} n}{2.3\ldots 6}$$
$$-\tfrac{1}{30}\frac{\phi^{\text{VII}} n}{2.3\ldots 8} + \tfrac{5}{66}\frac{\phi^{\text{IX}} n}{2.3\ldots 10} - \ldots$$

The fractions $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, &c., are the celebrated* *numbers of Bernoulli.* When A is not otherwise known, it may be determined by an instance, as hereinbefore shown. When the series is convergent, A is the sum *ad infinitum,* by which it is sometimes known. Though $\phi 0$ is made the first term, any other commencement might have been chosen; the constant A, properly found, sets the whole right.

As an example, we find that

$$1+\frac{1}{1+r}+\ldots+\frac{1}{1+nr} = A+\frac{1}{r}\log(1+nr)+\tfrac{1}{2}\frac{1}{1+nr}$$
$$-\tfrac{1}{6}\frac{r}{2(1+nr)^2}+\tfrac{1}{30}\frac{r^3}{4(1+nr)^4}-\ldots$$

* Todhunter, *Hist. of Probability*, p. 65. This work will be very useful to students who want to make a sound and complete preparation for the profession of an actuary.

$$\text{Make } n=0, \text{ and } A=\tfrac{1}{2}+\tfrac{1}{6}\frac{r}{2}-\tfrac{1}{30}\frac{r^3}{4}+\ldots$$

Hence the value of n years annuity at simple interest, r per £1, is

$$\frac{1}{1+r}+\ldots+\frac{1}{1+nr}=\frac{1}{r}\log(1+nr)-\tfrac{1}{2}\left(1-\frac{1}{1+nr}\right)+\tfrac{1}{6}\frac{r}{2}\left(1-\frac{1}{(1+nr)^2}\right)$$
$$-\tfrac{1}{30}\frac{r^3}{4}\left(1-\frac{1}{(1+nr)^4}\right)+\tfrac{1}{42}\frac{r^5}{6}\left(1-\frac{1}{(1+nr)^6}\right)-\tfrac{1}{30}\frac{r^7}{8}\left(1-\frac{1}{(1+nr)^8}\right)$$
$$+\tfrac{5}{66}\frac{r^9}{10}\left(1-\frac{1}{(1+nr)^{10}}\right).$$

Multiply by r, for r write n^{-1}, and we have

$$\frac{1}{n+1}+\ldots+\frac{1}{2n}=\log 2-\frac{1}{4n}+\tfrac{1}{6}\frac{1}{2n^2}\left(1-\frac{1}{2^2}\right)-\tfrac{1}{30}\frac{1}{4n^4}\left(1-\frac{1}{2^4}\right)+\ldots$$

Add this to the formula already found for $1+\ldots+\frac{1}{n}$, and we see, as it ought to be, that the result is that formula with n changed into $2n$.

On a Table for the Formation of Logarithms and Anti-Logarithms to Twelve Places. *By* Peter Gray, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

Part IV.—*History of the Method.*

So far as I am aware, the earliest method for the formation of logarithms, in which it was proposed to resolve the number whose logarithm is required into factors *by a continuous process*, is one which was published by Mr. Manning in the *Philosophical Transactions* for 1806. My acquaintance with Mr. Manning's method is derived, not from the original paper, which I have not seen, but from a reprint of it, "nearly as it stands," in Young's *Elementary Essay on the Computation of Logarithms.** Mr. Manning applies his method to the formation of Naperian logarithms only; but it is equally applicable to the formation of common logarithms. I repeat, in accordance with this method, an example I have already more than once given.

Required the logarithm of π.

The following is the process:—

* Second edition, London, 1835, pp. 67–79.

π=3·141592653590	÷2
1·57079632679~~5~~	
157079632680	
1·41371669411~~5~~	
141371669412	
1·27234502470~~3~~	
127234502470	
1·14511052223~~3~~	
114511052223	
1·0305994700~~1~~0	4
10305994700	
1·0202934753~~1~~0	
10202934753	
1·010090540~~5~~57	2
1010090541	
1·009080450~~0~~16	
1009080450	
1·008071369~~5~~66	
1008071370	
1·007063298~~1~~96	
1007063298	
1·006056234~~8~~98	
1006056235	
1·005050178~~6~~63	
1005050179	
1·004045128~~4~~84	
1004045128	
1·003041083~~3~~56	
1003041083	
1·002038042173	

1·002038042~~2~~73	
1002038042	
1·001036004~~2~~31	
1001036004	
1·0000349~~6~~8227	10
10000350	
1·0000249~~6~~7877	
10000250	
1·0000149~~6~~7627	
10000150	
1·000004~~9~~67477	3
1000005	
1·000003~~9~~67472	
1000004	
1·000002~~9~~67468	
1000003	
1·000001~~9~~67465	
1000002	
1·00000~~0~~967463	4

301029995664	log 2	
183029962243	4 Col.	1
8729610805	2 "	2
4345117740	10 "	3
13028900	3 "	5
1737179	4 "	6
390865	9 "	7
26058	6 "	8
3040	7 "	9
174	4 "	10
26	6 "	11
1	3 "	12
0·497149872695	=log π.	

The resolving process here, after preparation of the number as in the method already exemplified, consists obviously of series of subtractions, in each of which the subtrahend bears a definite and easily recognisable ratio to the minuend. In the first series, consisting of *four* subtractions (as indicated in the margin), this

ratio is 1 : 10; in the second (*two* subtractions), the ratio is 1 : 100; in the third (*ten** subtractions), 1 : 1,000; in the fourth (*three* subtractions), 1 : 100,000; and in the fifth (*four* subtractions), 1 : 1,000,000. The gradation amongst the ratios will be observed. The ratio 1 : 10,000 is missing; and the omission will be found to correspond to the passing over of a column in the subsequent part of the operation.

The object aimed at in the successive subtractions is the exhaustion of the decimal portion of the minuend, and the consequent reduction of the last remainder to unity. A subtrahend, therefore, greater than this decimal portion, is inadmissible; and it is when the next subtrahend would be in this condition that the necessity for a change in the ratio arises. And the point at which the change is requisite is indicated also usually but not universally (see in the example the change from 1 : 100 to 1 : 1,000) by the exhaustion, by the immediately preceding subtraction, of the figure occupying the highest place in the number to be exhausted.

When the first half of the decimal portion has been exhausted, that is, when the minuend has been reduced to unity followed by six ciphers, it is unnecessary to carry the process further, as the results we should obtain are already visible. The ratios of the several subtrahends to their minuends are those that follow by natural sequence the ratios already noted, and the number of subtractions in the several sums are indicated respectively by the figures composing the unexhausted portion of the minuend.

The second part of the operation shows the formation of the logarithm of π. The marginal numbers and their continuation in the last remainder form the arguments for the several columns; and the corresponding results, added to the logarithm of the preparing number, give the logarithm required.

The rationale of the foregoing process is as follows:—To subtract from a number one-tenth of itself is equivalent to multiplication of it by $\frac{9}{10}$. In like manner, subtraction of $\frac{1}{100}$ is equivalent to multiplication by $\frac{99}{100}$, and so on. But such multiplications are also equivalent to divisions by the reciprocals of the several multipliers respectively. It hence appears that the operations performed on π, in the example before us, are in effect one division by 2, four by $\frac{10}{9}$, two by $\frac{100}{99}$, ten by $\frac{1000}{999}$, and so on, the last

* Mr. Manning (or Mr. Young) says, that the number of subtractions in any series can "never exceed nine." Here, in an example casually taken, is evidence to the contrary.

quotient being unity, for the remainders are the successive quotients, as the process is now looked at. Hence we have the given number resolved into factors as follows:—

$$\pi=2\times\left(\frac{10}{9}\right)^{4}\times\left(\frac{100}{99}\right)^{2}\times\left(\frac{1000}{999}\right)^{10}\cdot\cdot\cdot\cdot\cdot$$

$$\therefore \log\pi=\log 2+4\log\frac{10}{9}+2\log\frac{100}{99}+10\log\frac{1000}{999}+\cdot\cdot\cdot\cdot$$

Hence a table* containing the first nine (or ten) multiples of $\log\frac{10}{9}$, $\log\frac{100}{99}$, &c., will furnish the logarithms requisite for the application of this method.

The preceding is the manner in which his method was presented by Mr. Manning. It admits of being presented in a different manner, as an application of a general principle, which includes also other methods, to be immediately described. The principle in question may be enunciated as follows:—

If, disregarding the decimal point, any given number be continuously multiplied by a series of factors till a result is obtained consisting of unity followed by n ciphers, the product of those factors will be the reciprocal of the given number, true to $n+1$ places.

Hence the sum of the logarithms of the factors will be the arithmetical complement of the logarithm of the given number; or, the sum of the logarithms of the reciprocals of the factors will be the logarithm of the given number.

The foregoing principle is too obvious to require a word to be said in explanation of it. In applying it, the point to be principally attended to is to arrange for the employment of factors which shall be easy to use, and whose logarithms can be readily formed and tabulated. In the foregoing example of Mr. Manning's process, the series of factors by which π is multiplied is

$$\tfrac{1}{2},\ \frac{9}{10}\,(4\text{ times}),\ \frac{99}{100}\,(\text{twice}),\ \frac{999}{1000}\,(10\text{ times}),\ \&\text{c.};$$

and the result is unity followed by twelve ciphers. Hence the sum of the cologarithms of these factors is the logarithm of π.

Mr. Manning's factors may be written as follows:—

* I have in my example *supposed* the data arranged in columns. They were not so arranged by Mr. Manning. And I may mention, that although I have here, as in previous examples, indicated a number of columns equal to that of the *possible* tabular entries, and shall follow the same course in subsequent examples, yet, in all cases with which we have had or shall have to do, the second half of the entries are obtained from a single column. (See note, p. 212.)

$$\frac{9}{10}=\frac{10-1}{10}=1-{\cdot}1,\ \frac{99}{100}=\frac{100-1}{100}=1-{\cdot}01,$$
$$\frac{999}{1000}=\frac{1000-1}{1000}=1-{\cdot}001,$$

and so on, the general form being $1-{\cdot}1^m$, or $1-{\cdot}1^m n$, where n is always unit. It does not appear whether Mr. Manning recognised the general principle above stated. Very probably he did not; but I think it is unquestionable that he did not put his factors into the form here given, since, had he done so, it is inconceivable that he should have failed to see that there is no necessity for the restriction of n to the value unity. To do him justice, however, he seems to have presented his method less as a *practical* method for the formation of logarithms generally, than as one for the formation of the fundamental logarithms preparatory to the construction of a complete table.

Mr. Manning's process, as we have seen, although beautifully simple, is also very laborious, and, for this reason, probably, it does not seem for many years to have attracted much attention, or to have undergone any improvement. At length, in Professor Young's reprint of Mr. Manning's paper, it came under the notice of Mr. Weddle, then of Newcastle, and subsequently of the Royal Military College, Sandhurst, who, in *The Mathematician* for Nov., 1845, pp. 17–25, reproduced it, with the improvement in the resolving process above hinted at, and also an extension to the converse problem of finding the number corresponding to a given logarithm. I repeat the example just given, worked according to Mr. Weddle's process.

π=3·141592653590	÷2
1·57079632679$\cancel{8}$	3
471238898039	
1·0995574287$\cancel{5}$6	9
98960168588	
1·00059726$\cancel{0}$168	05
500298630	
1·0000969$\cancel{0}$1538	9
90008726	
1·000006$\cancel{9}$52812	6
6000042	
1·00000$\cancel{0}$952770	

301029995664	log 2		
154901959986	3	Col.	1
40958607679	9	„	2
217201546	5	„	4
39088262	9	„	5
2605775	6	„	6
390865	9	„	7
21715	5	„	8
869	2	„	9
304	7	„	10
30	7	„	11
0	0	„	12
0·497149872695	log π		

The abbreviation here is manifest, the number of subtractions being reduced from twenty-three to five; and it will be found that it has not been purchased at an undue cost. The principle employed in the process is, as in Mr. Manning's process, that above enunciated, the given number being continuously multiplied till a product is attained consisting of unity followed by twelve ciphers.* The factors (after the first) are of the form $1-\cdot1^m n$, in which m takes the values 1, 2, 3, &c., successively, and n takes the values which in each case, when multiplied into the remainder, will give the greatest product not exceeding the decimal portion of that remainder. The operation consequently becomes assimilated to that of division, the dividend at any one point being the decimal portion of the remainder, the divisor the entire remainder, and the quotient a value of n.

π is here shown to be the reciprocal of the following product:—

$$\tfrac{1}{2}\times(1-\cdot3)$$
$$\times(1-\cdot09)$$
$$\times(1-\cdot0005),$$ and so on.

And hence the sum of the *cologarithms* of the component factors is the logarithm of π.

It is the cologarithms of these factors that Mr. Weddle tabulates,† and his anti-logarithmic method takes its form in accommodation to that of his tabulated values. I here present an example of it.

$\log\pi=$	0·497149872694				·000000376405	
ar. co.=	502850127306	log. 2			$999999\cancel{0}23595$	7
	301029995664				6999997	
	201820131642	3	Col.	1	$9999926\cancel{7}3598$	6
	154901959986				59999557	
	46918171656	9	,,	2	$99993262\cancel{4}041$	6
	40958607679				599959574	
	5959563977	1	,,	,,	$999332664\cancel{4}67$	3
	4364805402				2997997993	
	1594758575	3	,,	3	996334666474	1

* As in Mr. Manning's method, so also here, it is unnecessary formally to carry the resolving process beyond the exhaustion of the first six decimal places; since at that point what would be the result of its formal completion is visible in the remainder.

† He arranges them, not in columns, as the indications I have attached to the preceding example would seem to imply, but in an unbroken series. There are other breaches of continuity in the method as delivered by Mr. Weddle, which also I have removed.

1594758575	3	Col. 3
1304841688		
289916887	6	" 4
260654893		
29261994	6	" 5
26058451		
3203543	7	" 6
3040072		
163471	3	" 7
130288		
33183	7	" 8
30401		
2782	6	" 9
2606		
176	4	" 10
174		
2	5	" 12
2		

99633466664~~7~~4	1
9963346665	
9863713198~~0~~9	9
88773418782	
89759790102~~7~~	3
269279370308	
628318530719	÷2
3·141592653595 = π	

On the left is shown the decomposition of the arithmetical complement of the given logarithm into tabular logarithms, and ar-co. log π, or log $(1 \div \pi)$, is shown to be equal to

$$\log 2 + \log \frac{1}{1 - \cdot 3} + \log \frac{1}{1 - \cdot 09} + \log \frac{1}{1 - \cdot 01} + \ldots;$$

whence, taking the complements,

$$\log \pi = \log \tfrac{1}{2} + \log (1 - \cdot 3) + \log (1 - \cdot 09) \ldots$$

and therefore

$$\pi = \tfrac{1}{2} \times (1 - \cdot 3) \times (1 - \cdot 09) \times \ldots$$

The operation on the right shows the formation of π by the multiplication of the factors determined as above. It needs a little elucidation. The last six factors (Col. 11 is wanting) are $1 - \cdot 0000003$, $1 - \cdot 00000007$, &c., and their product, in the same mixed form, is $1 - \cdot 000000376405$, the negative portion of which therefore can be written down at once, by inspection.* Subtracting this from unity, we get for the same product, in a positive form, ·999999623595; and this being multiplied by the other factors in succession, the result is the value of π. The factors last referred

* See page 219.

to are used here in reverse order. They may, however, if we please, be used in direct order; in which case the compounding process would be in entire analogy—in fact identical—with the resolving process.

Mr. Weddle's paper came under my notice in the early part of 1846, and I was immediately attracted by the method as there developed. I soon perceived that it admitted of several modifications, by which its efficiency and the facility of its application would be increased; and, in particular, I observed that by the formation of a new table, adapted to two-figure values of n, the number of tabular entries requisite in any particular case would be reduced one-half. I therefore constructed such a table, to twelve places; and I communicated an account of the method, with the new table and my other improvements, to the *Mechanics' Magazine.** My communication appeared in the numbers for October and November, 1846.

I repeat the last example, worked by my two-figure table:—

$\pi =$	3·1415 9265 3590	$\div 2$
	1·5707 9632 679$\cancel{5}$	36
	4712 3889 8039	
	995 5742 8756	
	942 4777 9608	
	1·0053 0964 9$\cancel{1}$48	52
	50 2654 8246	
	2 8310 0902	
	2 0106 1930	
	1·0000 820$\cancel{3}$ 8972	82
	8000 6563	
	203 2409	
	200 0164	
	03 2245	

3010 2999 5664	log 2
1938 2002 6016	36 Col. 1
22 6422 3402	52 „ 2
3561 3608	82 „ 3
1 3029	03 „ 4
955	22 „ 5
20	45 „ 6
0·4971 4987 2694	$= \log \pi$

The above needs little explanation. The quotients being extended to two figures, π is resolved into the continued product following:—

* I am glad to have an opportunity of placing it upon record that Mr. Weddle, with whom I was not previously acquainted, called upon me and thanked me very cordially for the improvements I had introduced in his method, and for my endeavours to bring it more prominently before the public.

$$
\begin{aligned}
2 &\times (1-{\cdot}36)^{-1} \\
&\times (1-{\cdot}00{,}52)^{-1} \\
&\times (1-{\cdot}00{,}00{,}82)^{-1} \\
&\times (1-{\cdot}00{,}00{,}00{,}03)^{-1} \\
&\times (1-{\cdot}00{,}00{,}00{,}00{,}22)^{-1} \\
&\times (1-{\cdot}00{,}00{,}00{,}00{,}00{,}45)^{-1};
\end{aligned}
$$

and the logarithms of the factors composing it being taken from the tables, their sum is the logarithm required. The abbreviation arising from the use of the extended table is chiefly apparent in the second part of the operation, as comparison with the corresponding operation on p. 256 will show.

I now exhibit the anti-logarithmic process :—

log π =	4971 4987 2694		0000 0017 8739	
ar. co. =	5028 5012 7306	log 2	9999 9982 126$\cancel{1}$	37
	3010 2999 5664		2999 9994 6378	
	2018 2013 1642	37 Col. 1	6999 9987 4883	
	2006 5945 0546		699 9998 7488	
	11 6068 1096	26 " 2	6299 9988 7$\cancel{3}$95	26
	11 3063 6118		12 5999 9775	
	3004 4978	69 " 3	87 3988 7620	
	2996 7353		3 7799 9932	
	7 7625	17 " 4	6283 618$\cancel{8}$ 7688	69
	7 3830		3770 1713	
	3795	87 " 5	2418 5975	
	3778		565 5257	
	17	39 " 6	6283 1853 0718	÷2
	17			
	—		3·1415 9265 3590	$=\pi$

On the left is the decomposition of colog π into tabular values, which being the co-logarithms of $\frac{1}{2}$, $(1-{\cdot}37)$, $(1-{\cdot}00{,}26)$, $(1-{\cdot}00{,}00{,}69)$, &c., multiplication of these factors, as on the right, gives π. It will be perceived that the product of the last three factors is formed as in the last example, and the result multiplied by the remaining factors in *direct* order.

The operation now is a very compact one, and it might be rendered a little more so by the formation of a three-figure table adapted to it. It has its weak points, however. These are, first, that the multipliers occasionally *overlap* each other, rendering two entries in the same column necessary, as in the example on pp. 257, 258;

and secondly, that the anti-logarithmic operation partakes somewhat of an indirect character. But these are only slight blemishes; and had not another method which is free from them been found, I think there is no doubt that, had time been allowed, Mr. Weddle's would have been received as the best practical method that had as yet been devised for the formation of logarithms and anti-logarithms when more places are wanted than are afforded by the common tables. I proceed to give an account of the origination of the method now referred to, which is that which I have sought to develop, and to which the accompanying tables are adapted.

In the *Mathematician* for March, 1847, appeared a paper by Mr. Hearn, of the Royal Military College, Sandhurst, intituled, "Practical method of forming logarithms and anti-logarithms, independently of extensive tables." Mr. Hearn's logarithmic method, as described in his paper, was identical with Mr. Weddle's.* It, therefore, needs no further notice here. His anti-logarithmic method, however, is altogether different. It is the one that I have appropriated; and the example on pp. 132, 133, may be accepted as an illustration of it, although Mr. Hearn's table extended to only ten places, and the operation as here given is considerably systematized.

In the *Mechanics' Magazine* for February 12th, 1848, I gave an account and exemplification of Mr. Hearn's anti-logarithmic method, and also a table, to twelve places, adapting it to the two-figure process. The advantage gained in point of abbreviation and facility by this extension of the data, will be appreciated on comparison of the example on pp. 132, 133, already referred to, with that on p. 135, which latter is worked by the table which accompanied my paper. My paper, I perceive, is dated December 21st, 1847, very soon after which date I discovered the applicability of the

* It was singular, to say the least, to find given as original, and without remark, a method which had appeared in the pages of the same periodical sixteen months before. I had some correspondence with Mr. Hearn on the subject of his paper, in the course of which he informed me of the circumstances connected with the preparation and publication of it. The account he gave me was confirmed by a friend of his own, a gentleman of unquestionable veracity; and also, subsequently, so far as they were cognisant of the circumstances, by the Editors of the *Mathematician*. The circumstances were as follows:—The paper was prepared not later than September, 1845, at least two months before the appearance of Mr. Weddle's paper. In November of the same year it was handed to one of the Editors of the *Mathematician*; and the identity of the logarithmic methods developed in the two papers having been recognised, Mr. Hearn accompanied it with the requisite explanation. For some reason unexplained the paper did not appear till March, 1847, and the explanation was not given. It was given subsequently, as already hinted; but the omission of it at the proper time exposed Mr. Hearn to a charge of plagiarism, which is now seen to have been altogether undeserved.

It is right to add that Mr. Weddle's paper is dated August 18, 1845; and that he states in it that his methods "were discovered nearly seven years ago."

table to the direct operation—that of forming the logarithms of numbers, and showed it to my friend, the late Mr. Woollgar, and others. On the 7th February following, in a private note, I informed Mr. Robertson, the Editor of the *Mechanics' Magazine*, of the discovery, and asked for space to explain it in his pages. In his reply, dated on the 14th, he stated that he had received a paper in which the adaptation of the table to which I referred was shown, and offered to publish along with that paper whatever I might have to communicate on the subject. I immediately, on the 15th, wrote to Mr. Robertson, communicating an example of my method, and both letters appeared in the Number of the Magazine for February 26th. Mine bore my own name, and the other had attached to it the initials W. O. The following are the examples of the two methods as they appeared in the papers to which reference has just been made. The first is mine, and the second is W. O.'s :—

	3·14,159,2,65359,0	3 × 104
	3 12	
3 1 2)	2 159	69
1 8 7 2	1 872	
	287 2	
2 8 0 8	280 8	
3 1 4 1 5 2 8)	6 4 65359	20
6 3	6 2 83056	
3,1,4,1,5,9,1)	1 82303 0	58
	1 57079 6	
	25223 4	
	25132 7	
	90 7	02
	62 8	
	27 9	89
	25 1	
	2 8	
	2 8	

477121254720	log 3
17033339299	04
2986340857	69
8685803	20
251891	58
87	02
39	89
·497149872696	=log. required.

3141,5926,5359,0	×3		
9424,7779,6077,0	06		
565,4866,7764,6		Log 3	=4771,2125,4720
9990,2646,3841,6	09	Col. 1 06=	253,0586,5265
8,9912,3816,5		Col. 2 09=	3,9068,9250
		Col. 3 74=	3213,6603
9999,2558,7658,1		Col. 4 41=	17,8061
6999,4791,1	74	Col. 5 78=	3388
399,9702,3		Col. 6 48=	21
9999,9958,2151,5			5028,5012,7308
41,7848,5	ar. com.	ar. com.	=4971,4987,2692=required log.

My process differs but little in form from that which I now use, as will be apparent on comparison of it with the two-figure examples on p. 128, with the first of which, indeed, it has much in common. It therefore needs no further explanation.

The second process—that of W. O.—depends, like those of Mr. Manning and Mr. Weddle, on the principle laid down on p. 255, the distinction being that the factors employed by W. O. are of the form $1+n$, in consequence of which additions take the place of subtractions. The proper multiplier in each case is the greatest number that can be employed, so that the sum of the multiplicand and the new product shall not exceed unity; it is, therefore, usually, after the first step, the complement to nine, of the first figure in the multiplicand that differs from nine. As in Mr. Manning's and Mr. Weddle's processes, the formal operation here ceases when the first half of the quotient figures (as I shall call them) have been determined, and the remaining half are the complementary figures of the latter portion of what would be the next multiplicand. The foregoing process being proposed for use with my two-figure table, which contains the logarithms of the factors employed, the sum of the tabular results corresponding to the several multipliers is the cologarithm of the given number, from which the logarithm can, of course, be immediately deduced.

In illustrating, as I have done, and presenting for adoption my own method, I have sufficiently indicated my preference of it to that of W. O. It is natural I should be partial to a method originating with myself; but it will be expected that I should be able to justify my preference on other grounds. And I believe I can do so. While in regard to efficiency and the number of

figures employed the two methods may be admitted to be on an equality, I claim for my own superiority in point of simplicity and facility. In it the resolving process differs but little from the already familiar process of ordinary division, becoming more closely assimilated to that process as the table is extended for the purpose of reducing the number of tabular entries. The corresponding process in W. O.'s method, on the other hand, is of a mixed character, requiring a considerable amount of care and attention for the avoidance of error; and it is in no degree improved by the extension of the table. All this will be sufficiently apparent if an example be worked, using the three-figure table. W. O.'s method also is somewhat indirect: it is the cologarithm of the given number that is first found, from which the logarithm has to be deduced.*

In an operation such as the present, the necessity for which arises only now and then, revision of the rules for its performance may occasionally be necessary when the operation presents itself. And for this reason, irrespective of the more obvious one, the greatest attainable simplicity in both the operation itself and the description of it is desirable.

The author of the method brought, as above, into comparison with mine—the owner of the initials W. O.—was the late William Orchard, well known in actuarial circles as the author of a useful and meritorious work on Assurance Premiums. He introduced himself to me just after the simultaneous publication of our methods in the *Mechanics' Magazine;* and we found when we met, that, besides the subject that had brought us together, there were a number of others in which we felt a common interest. Of those one to which Mr. Orchard had devoted much attention was the subject of life contingencies. He published his work above referred to in 1850, and in 1851 he became one of my coadjutors in the preparation of a work consisting of results deduced from the Carlisle table of mortality.†

Mr. Orchard died in 1854, at the age of thirty; and Mr. Weddle and Mr. Hearn having also died in their early prime, it has fallen to me, in the foregoing historical sketch, to endeavour to apportion to each his share in the origination and development

* This indirectness would be got rid of if the cologarithms instead of the logarithms of the factors were tabulated. But it would then appear in the converse process, as it does in Mr. Weddle's method. There is no avoiding it, in the one operation or the other, in any method making use of the principle laid down on p. 255.

† Further particulars regarding Mr. Orchard are contained in an Introduction, prepared by me, to a re-issue of his work in 1856.

of the methods which have culminated, as I conceive, in that which is presented for adoption in the preceding pages. I have sought to do so impartially, and to the best of my information and ability; and I hope it will not be found that in what I have written I have afforded cause for complaint to the friends of any of the gentlemen whose claims I have had to discuss.

It only remains now that I should mention, that in Captain (now Major-General) Shortrede's *Logarithmic Tables** (1849), all the methods I have exemplified (with the exception of Mr. Manning's) are illustrated; and one and two-figure tables, of both Weddle's and Hearn's forms, are given for their application—the two-figure tables extending to 16 places and the one-figure to 25.

On Government Annuity and Assurance Rates and Regulations. By MARCUS N. ADLER, M.A., *of the Alliance Assurance Office, Fellow of the Institute of Actuaries and of the Statistical Society.*

[Read before the Institute, the 24th April, 1865.]

IN briefly commenting upon the papers that have been presented to Parliament on the subject of life assurances and annuities now granted by Government, it is less my object to criticize the measures that have been adopted, than to explain the mode of construction of the several tables of rates that are being acted upon. In consequence of several assumptions and suppositions, these have not been computed in the mode usually adopted by actuaries.

I may remind the members, that last year, when the Bill which has now passed into law was before Parliament, I had the honour of reading a paper before the Institute on this subject, and it is a source of satisfaction to me, that most of the suggestions contained in that paper appear to have been adopted. I feel particularly gratified that an Act was passed towards the end of the last Session, the 27 & 28 Vict., cap. 56, by which Probates or Letters of Administration for amounts not exceeding one hundred pounds, are

* Will it be considered out of place if I here call the attention of computers to this very useful work? In a handy volume it contains, besides sundry smaller tables, the ordinary seven-figure logarithmic table, and also an anti-logarithmic table of like extent. The convenience of the latter table, when many numbers to logarithms of seven places have to be taken out, is hardly conceivable by those who have not made trial of it. It is a great advantage also to have both tables in a single volume. The price of the book is, I believe, only twelve shillings.

exempt from stamp duty,—a boon which will be much appreciated by the working classes.

The table of mortality, the rate of interest, the loading in the case of premiums payable more than once a year, and the whole mode in which the tables are presented, are precisely those I ventured to suggest last year; and no doubt the members will share my opinion, that in the case of surrender of policies the authorities are acting prudently in not undertaking to return the whole or half the premium, as was originally proposed, but an amount depending upon the duration of the policy, the age at entry, and other elements. I shall have occasion to revert to the question of surrenders, but I may here at once express the hope, that the publication of the tables for surrender values will not be deferred. Although policies may not actually attain a value until they have been five years in force—that is, not before the year 1870 at the earliest —still, as it is intended to publish the tables of surrender values, it would be well if it were done at once, so that intending assurers may know exactly what bargain they enter into with Government.

To proceed to the consideration of the tables themselves.

As regards annuities—the old Government tables, founded on Mr. Finlaison's observations made, in the year 1823, upon the males and females of the tontines and sinking fund, are made use of; in fact, the very tables are adopted which have hitherto been acted upon at the National Debt Office—the rate of interest being 3½ per cent.* The premiums for assurances are based upon the third English life table, lately published by Dr. Farr, the rate of interest being taken at 3 per cent. Comparing the two tables, we should find that if the third English life table were adopted throughout, the rates for annuities would be somewhat higher than they are at present, and such proceeding would be justified by the well-known longevity of annuitants. True, the rates now charged are based upon the mortality experienced by that very class of annuitants; but we know that the value of life is increasing, and besides the fact of allowing 3½ per cent. interest, instead of 3 per cent., is a sufficient advantage in itself to annuitants. But more: the tables for the grant of deferred annuities and deferred monthly allowances (page 26 and 27 of the tables), which have been calculated on purpose to go hand in hand with the insurance rates, have a loading of 10 per cent. upon the net rates. The pure premiums for assurances are loaded 20 per cent. Now, it may be questioned

* It may be interesting to know that the rates granted under 3 Wm. IV., cap. 14, were computed at 3¾ per cent., and that no distinction was then made between the sexes.

whether such an arrangement is equitable. In the case of annuities, the trouble which the loading is intended to cover, is about as great again as in the case of assurances. For not only has the Office to *collect* the premiums in small instalments, but it has to *pay* the money away in small weekly or monthly stipends, whereas in assurance transactions, the amount contracted for is, in all cases, paid away in one sum.

The tables for the granting of Government life annuities were constructed on the assumption that the annuities were payable half-yearly, the first payment becoming due on the second quarterly day of payment next following the date of purchase. It is found by experience, that the public, in order to have the utmost benefit for their money, purchase these annuities but a few days before the respective quarter days, receiving the first payment of the annuity but a little after a quarter of a year's interval. The interest, as in the case of the dividends upon the public funds, is considered convertible half-yearly, so that half-yearly life tables had to be constructed, in the manner sketched out at page 334 in Griffith Davies' *Treatise on Annuities*, and analogous to those given by Dr. Farr in his recent work.

Of this fact Mr. J. W. Stephenson, the writer of a letter in the last Number of the *Assurance Magazine*, does not appear to be aware; and yet it must be admitted that this procedure is most conformable to practice, as, in the majority of cases, both interest and annuities are payable half-yearly. The formula deduced by Mr. Stephenson for the single premium of a deferred annuity, with the condition that the premiums paid are to be returnable on death or on application, at any time prior to the commencement of the annuity, is only partially correct, for the option of the withdrawal of the premium does not appear to have been duly taken into account. Mr. Stephenson assumes that the interest of the single or annual premium is to be applied year by year during the term to the purchase of an annuity deferred n years. The computations for Government were made upon a different supposition. The Act 16 & 17 Vict., cap. 45, sec. 4, provides, that deferred life annuities may be granted on condition that the purchase-money shall be returnable, but without interest. Acting in accordance with this law, the Government premiums for annuities, "money returnable," were computed by a mode which does not appear to have been known hitherto outside the doors of the National Debt Office, thus:—In the case of an annuity to a person now aged x years, such annuity to commence

n years hence, the single premium for an immediate annuity to a life aged $\overline{x+n}$ years discounted for n years was taken as the single premium, and this amount divided by the present value of £1 per annum for n years, the first payment due immediately, gave the annual premium.

It is not for me to justify a course that was duly approved of and has been acted upon for so many years—for, be it remembered, that these annuities were granted also under 3 Wm. IV., cap. 14, and surely the author of the tables would know the full import and bearings of the words of the Act, in the drawing up of which he was doubtlessly consulted. If interest were taken into account in the manner pointed out by Mr. Stephenson, the National Debt Office would give the full benefit of interest at 3½ per cent. to those who deposit money under the form of an annuity, whereas interest to Friendly Societies is now only allowed at the rate of 3 per cent., and to depositors with Post Office Savings Banks at little more than 2½ per cent., an arrangement which would clearly be less equitable than that adopted by Government.

Mr. Stephenson further draws attention to the fact that, according to the Government tables the value of life at age 31 is greater than that at age 21; and similarly, that the value of an annuity for 10 years is precisely the same at the ages of 21 and 39. It was hardly necessary to deduce these apparent anomalies from a comparison of the tables of immediate and deferred annuities; a mere glance at Observation 20, page 58, of Mr. John Finlaison's Report on the Law of Mortality of the Government Life Annuitants, would suffice to enable one to notice there a like phenomenon. The mortality per cent. of the males of the Tontines and Sinking Fund, at age 13, is ·52586; from this point it increases to age 24, at which the mortality is 1·41539; then it diminishes up to age 34, at which it is 1·24006, and it is only about the age of 48 that the same probability of death is exhibited as at age 24.

In the words of Sir John Lubbock,* "observations such as those exhibited by Mr. Finlaison, where the deaths are given at every age, are particularly well calculated to determine delicate points, such as any small increase of the rate of mortality at different ages."

But the experience table of the Government male annuitants is far from being the only table presenting this peculiarity. Similar aberrations are found to exist in Dr. Heym's experience table of

* *Transactions of the Cambridge Philosophical Society.*

the males of the Berlin Widows' Fund;* in De Montferrand's table of mortality of males in France;† in Quetelet's tables founded on the statistical returns of Belgium,‡ &c.

Professor Buchanan has also alluded to this subject in a paper read before the British Association in the year 1855. He quotes Quetelet as ascribing the inordinate mortality at age 23 and 24 to the violence of the passions at that age. The same results occur among females, although obscured by the increased mortality among them at a later age, from dangers peculiar to the sex.

The peculiarity adverted to did not even escape Dr. Price, nearly a century ago. In speaking of the Northampton Tables, he says,§ "The Bills give the numbers dying annually between 20 and 30 greater than between 30 and 40; but this being a circumstance which does not exist in any other register of mortality, and undoubtedly owing to some accidental and local causes, the decrements were made equal between 22 and 40; preserving, however, the total of deaths between 20 and 40 the same that the Bills gave them."

Here we plainly see one of the disadvantages of graduating a table of mortality in too sweeping and arbitrary a manner; it will always be found a proceeding calculated to efface any peculiarities exhibited by the original observations.

From the examples above adduced, I trust I have succeeded in showing that the circumstance alluded to *does* exist in other tables of mortality, and being exhibited not only in English but in foreign tables also, it can be ascribed neither to accidental nor to local causes.

It seems then to be abundantly proved that the mortality does not increase from age to age; and so far from being startled at Mr. Stephenson's discovery in connexion with the Government annuity tables, we may well be surprised at that gentleman not having been acquainted with so noteworthy a circumstance, upon which I only fear I have dwelt at too great a length. I now pass to the consideration of the rates for assurance.

In the statement accompanying the tables it is remarked, that, as far as practicable, the addition to the net premium for costs and charges was made proportional to the work that was to be done in return for it. Generally speaking, the loading is charged not only for the trouble of collecting the premiums and for Office expenses, but

* *Rundschau der Versicherungen*, jahrgang iv., p. 291.
† *Journal de l'Ecole Royale Polytechnique*, tom xvi., p. 306.
‡ *Bulletin de la Commission Centrale*, vol. v.
§ *Observations on Reversionary Payments*, 5th edition, vol. i., p. 352.

also to cover the risk of fluctuation. This latter seems to be altogether ignored in the tables; but, on the other hand, by 27 & 28 Vict., cap. 46, s. 3, provision is made, that if from the quinquennial valuation it shall appear, that the amount of liabilities is greater than that of the assets, the deficiency shall be made good from the Consolidated Fund. But if it shall appear that the value of the assets is more than sufficient to discharge the liabilities, an amount, not exceeding four-fifths of the surplus, shall be applied to the cancelling of the securities held by the Commissioners for the Reduction of the National Debt.* Nothing is said about the remaining one-fifth, but it may be presumed that it is intended to serve as a kind of reserve fund.

It follows, at all events, that the loading is only meant to cover charges and expenses, and under those circumstances it appears to be rather unfairly levied. In cases where the premium is paid more than once a year, be it half-yearly, quarterly, monthly, or weekly, there is an equal loading—20 per cent.; but in the case of the premiums payable yearly, the loading is but 10 per cent. Now, this cannot but be regarded as an injustice to that large class of intending assurers, to whom it would be more convenient to pay the premiums half-yearly or quarterly than yearly. How many are there who have to await the payment of their half-yearly dividends, small though they be, or of their quarterly salaries, before they are in a position to pay their premiums? Now, why should these people be charged nearly 13 per cent. more than those who pay their premiums yearly, and be placed upon the same footing with those from whom the premiums have to be collected monthly or weekly? The exact addition for the chance of death and loss of interest in the case of half-yearly premiums is under 2½ per cent. Even taking the most unfavourable view—viz., that adopted by Dr. Farr—which I shall have occasion to allude to further on, the addition does not exceed 3 per cent. No respectable Assurance Company would charge much more than this. As the matter at present stands with regard to the Government rates for assurance, the difference between the yearly and half-yearly premiums range from 4*s*. to £1 for every £100 assured.

In the construction of the tables the assumption has been made throughout, that the amount assured is not payable at the end of the year in which the assured dies, but at the instant of death. I

* According to the wording of the Act, at the end of every five years a valuation of the engagements made and liabilities incurred during the preceding five years is to be prepared. Surely it cannot be intended to ignore all the transactions entered into prior to the five years preceding the valuation.

will not stop to inquire whether $\sqrt{1+i}$ or $1+\frac{i}{2}$ should be the proper multiplier to transform the present value of a sum payable at the end of the year of death into that payable at the instant of death. Perhaps it may be recollected, that this point was fully discussed in the pages of the *Assurance Magazine* some years ago. No one will however dispute that, theoretically, the exact value for an assurance payable at the instant of death is that given by Baily—viz., $\frac{i}{\log_e (1+i)} A_x$, where A_x is the present value of a reversion payable at the end of the year of death. Expanding this expression, we have $\left(1+\frac{i}{2}-\frac{i^2}{12}+\ldots\right)A_x$, or $\left(1+\frac{i}{2}\right)A_x$ nearly, which has been adopted by Dr. Farr.

Tables 1 and 2, showing what single premium or consideration money, in one sum, must be paid in any year to assure the payment of a certain sum at death, have been calculated on this assumption, that is, the value of the reversion payable at the end of the year of death, at 3 per cent., has been multiplied by 1·015. This net premium was then loaded by 2 per cent., and a further addition of 2*s.* to the single premiums made, in all cases of insurances for sums under £50.

It may be asked who will be willing to pay the Government a sum of money as a single premium, when by investing it in Consols, or in some other security, and applying the interest towards the payment of the yearly or half-yearly premium of an insurance, a person can secure the same, and even greater benefits, without at all parting with his capital? For instance, a person aged 20 years next birthday, instead of paying to the Government Assurance Office the sum of £37. 0*s.* 2*d.*, as a single premium for an assurance of £100, would do much better to take therefrom £1. 4*s.*, as the first annual premium for an assurance of £66, and invest the balance in Consols. The dividend accruing from this investment would suffice to keep the insurance on foot. Whilst retaining nearly £36 during his lifetime, a person would at his death bequeath to his heirs the sum of £102—viz., £36 invested, together with £66, the amount assured.

True, if arrangements were made by which the assured under Tables 1 and 2 could receive back the amount paid by him, whenever he wishes to surrender the policy, the rates proposed would not be found too high; but no such terms appear to have yet been offered. Surely then the condition that no policy or contract

for assurance will have attained a surrender value until it has been 5 years in force, that is, not until the year 1870, ought not to hold in these cases. Considerations for the surrender of single premium policies should be given at any time, even immediately after the completion of the contract.

Table 3, showing the annual premium that must be paid for an assurance for the whole of life, is at once deduced from the table of the net annual premiums given in page 52 of Dr. Farr's work, by multiplication by 1·015 and an addition of 10 per cent.

Table 4, showing what sum payable at death may be assured by the payment of an annual premium of £1, is merely the reciprocal of the values in Table 3.

We now pass to the tables showing the amount assured when the premiums are paid more than once a year. Here the assumption is made that the premiums are paid "continuously," by which is meant that the year is considered to be divided into a number of small intervals, and a proportional part of the payment is made at each of these intervals. It is also assumed that the numbers living through the year are represented by the numbers living in the middle of the year, that the deaths are equally distributed over the year, and that the average time of the investment of the premiums received, may be set down at half a year, as the deposits are equally made throughout the year.

On these assumptions Dr. Farr gives, in the Appendix to the 12th Report of the Registrar-General, the different formulæ, and deduces that π'_x, the annual premium for an assurance at the death of a person now aged x years, $=\dfrac{M'_x}{Q'_x}$,

$$\text{where } M'_x=\left(1+\frac{i}{2}\right)M_x,$$

$$\text{and } Q'_x=P'_x+P'_{x+1}+\ldots.=\left(1+\frac{i}{2}\right)(l_{x+\frac{1}{2}}v^{x+1}+l_{x+1\frac{1}{2}}v^{x+2}+\ldots.).$$

Both M'_x and Q'_x, which latter resembles the ordinary N_x of the commutation tables, are tabulated at page cl. of the Preface to the English Life Tables.

Similarly, where only a limited number of premiums are paid, such as is assumed in Tables 6 and 7, the formula is $\dfrac{M'_x}{Q'_x-Q'_{x+n}}$.

I have before remarked on the heavy addition in respect of premiums payable more than once a year. A person, however, who can manage to pay a somewhat high amount in the first instance, viz., one whole year's premium, can evade all this extra

expense, without giving any the less trouble to the Post Office authorities. Anyone can open an account with the Post Office Savings Bank by paying into it, weekly or monthly, his small savings—be they as low as 1*s*. At the end of each year he can direct, by virtue of § 33 of the Government Insurance Regulations, that the requisite amount of annual premium should be placed to the credit of his life assurance account. A depositor would thereby not only save the increased rate of premium, but realize besides the usual interest of 6*d*. per £1 on his investment with the Post Office Savings Bank.

The Government propose to exclude persons following the occupations of miners, butchers, innkeepers, sailors or mariners from the benefit of life assurance at the ordinary terms. It may be presumed that these occupations are so singled out, because in the 14th Report of the Registrar-General, persons following these occupations exhibit the greatest mortality. It must be regretted that such scanty information is afforded on this head in the different Reports of the Registrar-General, but there is little doubt that under the able administration of that department the results of the last census will be made available for this interesting inquiry.

The following table exhibits the mortality of sundry occupations given in the 14th Report of the Registrar-General, as compared with that deduced by Mr. Henry Ratcliffe.

Comparative Annual Mortality per Cent. according to Occupations.

Ages.	Dr. Farr's Observations in 1851.			Mr. Ratcliffe's Observations on Friendly Societies, 1850.				
	English Males.	Publicans.	Miners.	Miners.	Clerks.	Printers.	Potters.	Total Experience of Manchester Unity.
25–35	·948	1·383	·849	·899	1·065	·646	1·267	·830
35–45	1·236	2·045	1·135	1·221	1·289	1·244	1·655	1·053
45–55	1·787	2·834	2·015	2·313	4·377	4·046	2·697	1·753
55–65	3·031	3·897	3·450	3·966	5·514	6·172	3·790	3·531
65–75	6·396	8·151	8·051	8·334	7·799	7·038	6·384	6·623
75–85	14·055	18·084	17·867	11·406	12·686	11·007	10·719	12·584

The discrepancies between the two investigations are, no doubt, great in some cases; but we must recollect that Dr. Farr gives the mortality of all who compose a certain class, whereas Mr. Radcliffe gives only the mortality of such of those classes who may have joined Friendly Societies. Most of Mr. Radcliffe's results are fully borne out by Mr. Neison's observations, given in his work on *Vital Statistics*.

It will be seen that the rate of mortality of clerks, printers, and potters is almost throughout higher than that of miners; and whereas one-half of the whole of the persons forming Mr. Radcliffe's experience entering at age 18 die off before attaining 63½ years, one-half of the class of clerks entering at 18 die off before attaining age 54, one-half of the class of printers before age 56, and potters before age 57, whereas one-half of the number of miners die off only at 60. Cutlers, plumbers, coopers, stonemasons, &c., also exhibit a more unfavourable mortality than miners.

On the other hand, the class of butchers, agreeably with the investigations of Messrs. Radcliffe and Neison contrasts rather favourably with the mortality of other trades; and this view is confirmed in various Reports of Dr. Letheby on the sanitary condition of the city of London. Without stopping to pursue this inquiry further, I will only add that Dr. Thomson,* Dr. Ward,† and the author of an article in the *Quarterly Review* for January, 1860, seem also to dissent in some measure from the views propounded in the 14th Registrar-General's Report.

Again, we know that in certain localities and towns in the United Kingdom the mortality is much above the average. Dr. Letheby considers that a mortality of 15 in the 1,000 is that which is natural in this country, whereas the rate in the *city of London* is 30 in the 1,000, or as high again. The accompanying table exhibits the mortality in various localities in this country as deduced from several tables.

Comparative Annual Mortality per Cent. according to Locality.

Age.	English Life Table, No. 3. Dr. Farr.	Healthy English Districts. Dr. Farr.	Scotland. Neison.	Glasgow. Ratcliffe.	Friendly Societies' Experience.				Age.
					Finlaison.	Neison.			
						Rural.	Towns.	Cities.	
20	·846	·728	·701	·852	·74	·739	·535	·645	20
30	1·038	·856	·793	1·769	·77	·711	·740	·928	30
40	1·295	·963	1·077	2·535	1·03	·797	·960	1·401	40
50	1·768	1·232	1·583	3·600	1·50	1·200	1·627	1·940	50
60	3·116	2·228	2·910	8·013	2·61	2·160	3·273	3·046	60
70	6·104	5·239	5·095	..	..	5·044	7·228	6·392	70

If an extra rate be charged in respect of those who follow hazardous occupations, why should not a similar additional charge

* "On the Influence of Occupation on Health and Life;" a paper read by Dr. E. H. Thompson, at the Social Science Congress for 1862.

† "On the Medical Estimate for Life Assurance;" by Dr. H. Ward. (*Assurance Magazine*, vol. viii.)

be made in respect of those who are unlucky enough to live in unhealthy districts?

One other matter I wish to refer to before I conclude. Although I have from the beginning given my cordial support to the measure which has now passed into law, I did so feeling persuaded that the respectable Assurance Companies had no cause to fear Government competition. My presumptions in this respect have been so far borne out. The premiums charged by Government are higher than the non-participating rates of most of the respectable Assurance Companies. The latter also offer that great advantage to the assured, that in the event of his engaging at any future time in certain hazardous trades he is not liable to an extra payment. Assurance Companies will, perhaps, in the long run be the gainers, for Government is in fact advertising for them, and an employer who recommends his servant to assure, can certainly not do less than assure his own life, only for a larger amount, and in order to do this he must necessarily apply to the Assurance Company.

I also supported the measure in the hope that by the extension of assurance in its most attractive and popular form, it would be largely made use of by the working classes, and thus confer a lasting benefit upon the country at large. Now, it is a mistake to suppose that the facilities in the modes of payment, or that the security the Government offers, will prove so attractive to working men as to induce them to abandon their clubs and friendly societies and to rush to assure with Government.

How was it with respect to annuities? Perhaps it is not generally known, that by the 3rd William IV., cap. 14, similar facilities, as are now offered by the new Act, were then granted. The consideration money in respect of annuities, immediate, deferred or temporary, could either be paid by weekly, monthly, quarterly, or yearly instalments, as suited the convenience of the purchaser. Yet so few availed themselves of this privilege, that the Act was repealed after having been 20 years in force.

As the case now stands with regard to assurances, I think the labouring men of this country will show a similar aversion to entering into contracts with Government direct, not because they are not provident—perhaps they are more provident than any other class of society—but because they are unwilling to sever their connexion with the clubs to which they are so attached. It may, under these circumstances, be worth considering that the present minimum of £20 for assurances be reduced, and that in the same manner as the General Post Office, agreeably with § 32 of the Regulations, is

empowered to make arrangements with superior officers and employers for collecting the premiums amongst those employed by them, so it should constitute all savings banks, but especially Friendly Societies, its agents.

No risk whatever would be incurred by Government, for the friendly societies would have to account weekly or monthly for the monies received, and the assured might be supplied with their receipts for the premiums paid direct from the Head Office. In this manner the greatest opponents to the Government scheme would become its warmest supporters, and our legislators, instead of administering a death blow to those ancient institutions, that are so entirely in harmony with the spirit of the nation, would preserve and render them a lasting service by freeing them of their greatest element of insecurity. I venture to think that by this step, more than by any other, the benevolent intentions of the promoters of the Government Assurance Bill will be most successfully realized.

On the Rates of Mortality and Marriage amongst Europeans in India. By Samuel Brown, F.S.S.

[Read before the British Association, Section (F), at Bath, September, 1864.]

In a paper which I had the honour of reading before the Institute of Actuaries in December, 1862, an inquiry was made into the rates of mortality and marriage amongst Europeans in India, but was principally confined to the experience amongst military officers, as recorded in the books of the Madras Military Fund, and compared with the records of similar funds in the other Presidencies. The data—which I was favoured with an opportunity of collecting during an elaborate investigation into the position and prospects of the fund —extended over the long period of fifty years, from 1808 to 1857, and related to more than 5,000 officers who had entered the fund in that period, and had either died, or withdrawn, or were living at the close of the observations, on the 1st January, 1858. The subdivision of the facts into two periods, of those who entered from 1808 to 1822, and from 1822 to 1857, showed a very marked diminution at every quinquennial period of age in the rate of mortality up to the age of 50, after which, in the latter period, the numbers were not sufficient fairly to carry on the comparison. On the average of all ages the rate in the former period was 3·92 per cent., and in the latter 2·69

per cent., though allowance must be made for the fact that some of the latter had not attained such advanced ages.

Another conclusion, clearly arrived at, was, that at all ages below 65 the mortality amongst married officers, was considerably less than amongst bachelors, being seldom more than 60 or 70 per cent. of the latter. The average rate of mortality at all ages was amongst bachelors 3·44 per cent., married officers 2·83 per cent., and widowers 4·45 per cent.

Amongst retired officers the rate of mortality in each class, whether bachelors, married men, or widowers, was found to be always highest at the younger ages, and to diminish with great regularity to the ages 55 to 60. After this age it seems to exceed generally, by about 25 per cent., the rate of mortality by Dr. Farr's healthy life table for males.

In regard to the marriage-rates, the observations were minute enough to afford some interesting deductions. A paper then recently read by Mr. Archibald Day, before the Institute of Actuaries, "On the Statistics of Marriages amongst the Families of the Peerage," extending for a period of 100 years preceding 31st December, 1855, and comprising the marriages amongst 2,721 bachelors, enabled me to make comparisons with those amongst military officers in the Madras army.

The marriages amongst the aristocracy, as compared with the general population, were observed to take place at a much later period of life, and still more so amongst widowers. This peculiarity was even more strongly observable amongst bachelors, being military officers in India, and to a certain extent amongst the widowers, the rates of marriage of the latter between 50 and 55 being double that of the peerage. It should be noticed that this period of life is nearly approaching that which shows the maximum rate of retirement, and may perhaps be connected in some way with the return of officers to Europe. Of officers in India, the average rate of marriage at all ages was 3·75 per cent. amongst the bachelors, and 9·25 per cent. amongst widowers, whilst in the peerage of Great Britain the rates were respectively 3·63 per cent. and 5·55 per cent.

In my former paper a table was also given, showing amongst 1,526 first marriages, 164 second, 17 third, and one fourth marriage, the number which were contracted amongst officers of the Madras army, at each quinquennial age of the husband with wives of the same or any other quinquennium of age, from which the conclusion was clearly drawn, that as the husband married later in life, the greater was the discrepancy of age between himself and his wife.

The average age at which bachelors marry appear to be about 30, and of their wives 23, a difference of about seven years. Under 20, the bachelors marry wives about three years younger than themselves, and from 20, the discrepancy in favour of a younger wife steadily increases each five years, till at 60–65 years of age for the husbands, it is twenty-five years younger for the wife. For widowers, the average age of marriage appears to be about 41, and their wives 27, a difference of fourteen years—just double that of bachelors. Under 25, in widowers' marriages, the average age of the wife is five years older than her husband, but afterwards the discrepancy of age in favour of a young wife increases till it is much higher than that of bachelors, being from 30 to 34 years when widowers marry at 60 and upwards.

Since the publication of that paper, the interest in the subject has by no means diminished. The vast impulse given to the commercial undertakings of India, railways, telegraph, and financial, land and trading companies, must have led to a great increase in the European population, an increase which, there is every reason to believe, will become annually greater as European skill and capital find profitable employment in those wide fields of enterprise. At the same time it will be many years before a proper organization can be given to the collection of the statistics of life and health amongst a population so scattered or migratory as this is likely to be. Young men, sent out as engineers for commercial, telegraph, and other companies, will, in most cases, have to proceed to new and unsettled districts, and frequently move about from one station to another. It would be well if all public companies, as well as the Government services, were to keep a careful register of their employés, their ages at admission, date of withdrawal or death, and, as far as possible, of their marriages, families, &c. In the meantime the most precise information on these subjects at present available will be found in the records of the annuity, provident, and pension funds for widows and children, which have been for many years established in the different Presidencies, both for the military and civil services of the Government. They require for their own purposes that the date of the birth, marriage, withdrawal, or death of each member should be furnished, as well as the date of birth of his wife, and of her second marriage or death, and of the births or death of all his children. If these registers had been as carefully kept from the beginning as they are at present, there would have accumulated by this time the materials for working out several important problems in population statistics. It is desirable, at any rate, to gather to-

gether what has hitherto been made public, for comparison with more complete data which may be given hereafter.

It was from the records of these funds that the facts stated in my former paper were drawn relative to the rates of mortality and marriage amongst military officers in India, and I propose now to extend the inquiry to Europeon civilians resident there.

The most important sources of original facts, in reference to the mortality and marriage amongst European civilians in India, will be found in the following reports, with some incidental notices in the publications which were enumerated in my former paper:—

In 1836 (20th February), Mr. Griffith Davies' "Report on the Bombay Civil Fund."

In 1842, Mr. Davies, for the Bengal Uncovenanted Civil Service Pension Fund, deduced a table of mortality from the Bengal Civil Service.

In 1850, "Report of the Committee on the Bengal Civil Service Fund."

In 1850, Mr. Davies' "Report on the Madras Civil Fund."

In 1851, Mr. Davies' "Report on the Bengal Civil Fund."

In 1852, Mr. Neison's "Report on the Bengal Civil Fund."

In 1852, Mr. Neison's "Report on the Madras Civil Fund."

In 1855, Mr. Neison's "Report on the Madras Civil Fund" (in which he introduces the mortality according to years of service).

In 1861 (26th June), Mr. W. Grant's "Report on the Subsidiary Branch of the Madras Civil Fund."

In 1861 (18th November), his "Report on the Charity Branch of the above Fund."

In 1861 (24th June), Mr. Neison's "Report on the Bombay Civil Provident Fund."

In addition to the above, the following reports on the medical funds contain much interesting matter:—

Madras Medical Fund, Mr. Neison's Reports, 16th February and 29th May, 1856.

Bombay Medical Fund, Mr. G. Davies' Report, 15th February, 1847.

Bombay Medical Fund, Mr. Neison's Reports, 2nd January, 1854, and 7th November, 1855.

Bengal Civilians.

In Mr. Davies' "Report on the Bengal Civil Fund," in June, 1851, he states that not having the means of forming mortality tables from their own experience, he had been obliged to examine the other Indian reports for original data to guide him. In 1842 he had formed for the Bengal Uncovenanted Service Family Pension Fund, a table of mortality amongst the Bengal Civil Service, from the lists of Dodwell and Miles. Such table gave the mortality below the age of 40, somewhat lower than the Northampton Table, and higher afterwards. But on examining Mr. Neison's table for the Bengal Military Fund, and considering that soon after the age of 40 the members of the Civil Service Fund begin to return to this country, he had determined to adopt the Northampton Table from the age of 40 and upwards, and continue it below that age by his own table from the Bengal Civil Service, above alluded to.

Mr. Neison, in his report of 14th December, 1852, following this, still regrets that he has not the actual experience of the Fund to refer to, but objects to the lists of Dodwell and Miles, which he considers worthless for the purpose of deducing the rates of mortality amongst the servants of the Company, since they were not compiled with this object in view, and can only be regarded as ordinary directories. Major Hannyngton, however, had pointed out a most important document, and one more trustworthy for the purpose. It is a "Register of the Honourable East India Company's Civil Servants of the Bengal Establishment, from 1790 to 1842, &c., compiled under the direction of the Honourable H. T. Prinsep, late Member of the Council of India, by Ramchunder Doss."

The rate of mortality for each quinquennial period of age, as given by the compiler in the introduction, and also the rates from an adjusted table deduced by Major Hannyngton, are as follow:—

Ages.	Exposed to Risk.	Died.	Rate per Cent.	Adjusted by Major Hannyngton. Rate per Cent.
20	231	8	3·47	2·55
21–25	4,782	93	1·95	1·98
26–30	4,010	84	2·09	1·83
31–35	3,177	48	1·51	2·04
36–40	2,172	60	2·76	2·45
41–45	1,496	44	2·94	3·09
46–50	818	29	3·55	3·82
51–55	392	23	5·87	4·55
56–60	152	5	3·30	5·31
61–65	57	3	5·26	6·20
66–70	14	1	7·20	7·47
71–75	2	..	..	9·15
76–80	..	..	..	11·63
81–85	..	..	..	17·86
86–90	..	..	..	24·83
91–95	..	..	..	37·93
96	..	..	..	100·
	17,302	398	2·30	3·38

But it had been clearly shown by the records of the military funds that the rate of mortality in India had diminished of late years, and, as the above table did not afford the means of a similar comparison, Mr. Neison recomputed the rates according to different decennial periods after the members' arrival from Europe. It is probable, however, that by this minute subdivision, the facts at some of the ages are too few to admit of averages for a fair comparison, and it will be quite sufficient to give Mr. Neison's rates for the two periods 1790–1819 and 1820–42:—

Bengal Civil Service—Members Arriving in India in the Years

Ages.	1790-1819.			1820-42.		
	Exposed to Risk.	Died.	Mortality per Cent.	Exposed to Risk.	Died.	Mortality per Cent.
21–25	2,898	51	1·76	2,006	41	2·04
26–30	2,550	50	1·96	1,528	30	1·96
31–35	2,248	40	1·78	975	10	1·03
36–40	1,937	48	2·48	285	4	1·40
	9,633	189	1·96	4,794	85	1·77

At the younger ages it would appear that the rate has somewhat increased, or remained the stationary, but at quinquennial ages above

30, the diminution in mortality is considerable, which Mr. Neison accounts for by supposing that persons of the most experience will be the first to take advantage of the precautions suggested as best calculated to preserve health. These observations, however, relate only to the members of the civil service, whilst actually employed in India. A very important question arises as to the rate of mortality amongst the members after retirement. Without the actual experience of the fund, which he thinks would be in this case so valuable, Mr. Neison argues, from the official documents to which he had access in the India House, relating to the retired officers of the Bengal Military Service, that the rate of mortality amongst them does not differ widely from that of the general population of England and Wales at corresponding ages; and further, that the rates of mortality amongst retired members, both of the civil and military services, are almost identical. He therefore constructed a new table for the valuation of the fund. Admitting that Mr. Davies' table, up to the age of 40, agreed very nearly with the ratio of deaths pointed out by the preceding facts, he had taken the same rates up to that age, but continued the table from the age of 45 by his own table, given in his report on the Bengal Military Fund in 1849; and between the ages 39 and 45, the terms were interpolated. The effect of this is to show at ages above 40, a considerable improvement in the duration of Indian lives, in fact to approximate after that period to the general rate of mortality in this country.

In the course of an investigation now proceeding into the Bengal Civil Fund, I have been favoured by the secretary not only with a table which enables me to bring down the observations of the Bengal Civil Service to a very recent date, but with the means of collecting the experience of the fund itself for the thirteen years 1850 to 1862 inclusive. These data are important, as they show the mortality of the members in each class, bachelors, married men, and widowers, as well as the mortality amongst females and children, and the rate of marriage amongst both sexes. The experience of the Bengal Civil Service has been divided into two periods, 1800 to 1830, and 1831 to 1858, showing at the middle ages, 20 to 40, a considerable diminution in the rate of mortality in the latter period.

The number who entered in the former period was 647, of these 283 died, 238 became annuitants, 60 withdrew, and 66 were living to 1859. The number exposed to risk was 13,887, and the rate of mortality 2·04 per cent. In the latter period 568 entered, of whom 96 died, 4 became annuitants, 26 withdrew, and 442

were living to 1859; the number exposed to risk was 5,631, and the rate of mortality 1·70 per cent.

The following table shows the rate of mortality at the quinquennial ages in the two periods referred to, and in the whole period 1801 to 1858 inclusive, compared also with Davies' table used in 1850 (which Mr. Neison also followed up to age 40), with Neison's table for the Bengal army, 1800 to 1847, and with Farr's healthy life table for males. The total rates per cent. also are given for the Bengal Civil Fund, thirteen years' experience, terminating 1st January, 1863, but the numbers are scarcely sufficient to divide the latter into the three classes, bachelors, married men, and widowers, except by grouping periods of 10 years of age together. The column for the Madras Civil Service, 1760 to 1853, is added from the comprehensive "Report on the Sanitary State of the Army in India," in which Dr. Farr has brought together such varied information from every available source.

Rates of Mortality per Cent.

Ages.	Bengal Civil Service. 1801-30.	1831-58.	Retired, 1801-58.	Active and Retired. Total, 1801-58.	Bengal Civil Fund. Thirteen Years to 1863.	Of whom were Killed in the Mutiny.	G. Davies. Used in 1850.	Neison. Bengal Army. 1800-47.	Madras Civil Service, 1760 to 1853.	Farr's Healthy Life Males.
14–	2·19	2·44	..	2·24	..	..	..	1·19	..	·72
20–	1·78	1·48	..	1·65	1·07	·41	1·41	2·19	1·40	·92
25–	2·20	1·73	..	2·02	1·80	·64	1·59	2·34	1·52	·99
30–	1·57	1·24	..	1·47	1·51	·44	1·77	2·62	1·55	·96
35–	2·00	2·81	8·00	2·19	1·88	·55	1·94	2·63	1·63	1·24
40–	2·02	3·01	1·08	2·08	1·08	·32	2·24	2·55	1·79	1·21
45–	2·19	..	1·14	1·76	2·16	·23	2·52	2·92	2·04	1·70
50–	4·29	..	1·81	2·54	2·46	·16	3·04	2·23	2·52	1·85
55–	3·77	..	2·55	2·71	2·07	..	3·59	2·54	2·84	2·86
60–	18·18	..	3·70	4·26	4·26	..	4·32	3·03	2·97	3·40
65–	..	..	6·25	6·22	7·79	..	5·43	1·52	3·57	5·71
70–	..	..	6·34	6·34	.	..	7·46	..	5·05	7·34
75–	..	..	20·93	20·93	14·29	..	10·81	..	8·50	12·59
80–	..	..	..	..	100·	..	..	..	..	..
	2·04	1·70	3·05	2·10	1·76	·41	..	2·28	..	
Exposed to Risk	13887·	5,631	3248·	22,766	7625·5	..	..	88,630		
Died ..	283·	96	99	478	134	31	..	2,019		

The records of the Bengal Civil Fund show the number of members killed in the mutiny, and the mortality is considerably

altered thereby, the large proportion of 31 out of 134 deaths being due to this cause, and principally affecting the ages 25 to 30, at which ages they amounted to nearly one-third of the total deaths; being all on active service, the ages above 55 remain unaltered.

From the thirteen years experience of the Bengal Civil Service Fund, I was enabled to trace the rates of mortality and marriage amongst bachelors and widowers, or of mortality and the chances of becoming widowers amongst married men at each age. The inquiry is too minute to be pursued here, but the summary may be given.

Of bachelors 213 were living on 1st January, 1850, and 386 entered since. Of these 14 were killed in the mutiny, 31 died, 23 withdrew, 245 married, 42 retired, and 244 were living on 1st January, 1863. The number exposed to risk was 2,919, of whom 1·54 per cent. died, including ·48 per cent. who were killed in the mutiny, and 8·39 (a very large proportion) married.

Of married men, 249 were living 1st January, 1850, of whom 29 were married to second wives; since then, 245 bachelors married, and 6 entered the fund as married men, 18 entered into a second marriage, and 1 for the third time. Of the total number 519, 60 died (of whom 15 were killed in the mutiny), 9 withdrew, 47 became widowers, 120 retired, and 283 were living 1st January, 1863. The number exposed to the risk was 3539·5, and the rate of mortality was 1·69 per cent., including ·42 per cent. killed.

Of widowers, 24 were living 1st January, 1850, and 47 became widowers since, of whom 9 died (2 of them being killed), 27 remarried (of whom 1 married for the third time), 14 retired, and 21 were living 1st January, 1863. The number exposed to risk was 274·5, and the rate of mortality was 3·28 per cent. (of whom ·73 per cent. were killed in the mutiny).

These facts all relate to service in India. The retired members might be traced in the same way under each class.

Marriage Rate.

The rates per cent. of marriage at each quinquennial age, both amongst bachelors and widowers, are very irregular, but they seem in nearly all cases to be unusually high. It would be well to continue the observations a few years longer. In the following table I have compared together the rates in the Bengal Civil Fund, in the Madras Military Fund, as given in my former paper, in the peerage of Great Britain, and in the general population of England and Wales, as shown in the Registrar-General's reports.

Rates per Cent. of Marriage.

Ages.	Bachelors.						Widowers.					
	Bengal Civil Fund, 1850–62 inclusive.		Madras Military Fund 1808–57 inclusive.		Peerage of Great Britain. (Day.)	General Population. (Farr.)	Bengal Civil Fund, 1850–62 inclusive.		Madras Military Fund, 1808–57 inclusive.		Peerage of Great Britain. (Day.)	General Population. (Farr.)
	Active.	Retired.	Active.	Retired.			Active.	Retired.	Active.	Retired.		
15–	—	—	·17	—	·19	·46	—	—	—	—	—	—
20–	8·16	—	2·61	·60	4·21	11·21	—	—	13·56	—	17·86	30·77
25–	8·74	—	4·82	3·48	7·70	12·21	22·64	—	11·45	—	14·91	35·79
30–	13·67	—	6·26	2·67	7·14	7·85	14·28	—	13·19	28·56	12·49	28·63
35–	7·50	—	5·40	2·67	5·47	4·56	13·68	—	8·66	40·	10·46	20·31
40–	2·60	14·29	4·06	2·94	3·95	2·80	11·11	—	9·90	17·39	9·95	14·08
45–	2·99	5·61	3·66	1·18	1·98	1·45	3·85	16·00	9·01	16·67	7·72	8·86
50–	3·92	1·67	4·85	1·51	1·07	·71	—	9·84	10·98	16·50	5·91	5·71
55–	—	12·50	3·49	1·20	1·05	·35	5·26	—	4·35	7·23	4·25	3·20
60–	—	—	—	—	—	·15	—	—	3·90	—	3·27	1·75
65–	—	—	—	—	—	·05	—	—	—	—	2·17	·86
70–	—	—	—	—	—	·03	—	—	3·70	—	1·81	·32
75–	—	—	—	—	—	·06	—	—	—	—	·83	·10
	8·39	4·44	3·75	2·23	3·63	—	9·84	5·85	9·25	13·71	5·55	—
Number living	2,919	135	45,439	2509·5	—	—	274·5	85·5	1,936	197·	—	—
Number who married ..	245	6	1,706	56	—	—	27·	5·	179	27·	—	—

From 244 cases in which the ages both of husband and wife were given, it may be concluded that the average age of a bachelor member of the fund on marrying is 28, and of his wife about 22; the difference is six years, and the age at marriage is a little below the age at which the bachelor members of the Madras Military Fund marry, which appears to be 30, and the wife 23, a difference of age of seven years.

In the following table is shown, for bachelors who marry at any quinquennial period of age, the number of wives at each quinquennial group of ages, whether older or younger than the husband.

Bengal Civil Fund. Bachelors Married in Thirteen Years, 1850-62 *inclusive.*

Age of Husband.	Age of Wife.					Total.	Per Cent. of Total Marriages, Age of Wife.					Total.
	17–	20–	25–	30–	35–40.		17–	20–	25–	30–	35–40.	
20–	26	53	7	1	..	87	10·6	21·7	2·9	·4	..	35·7
25–	18	47	11	2	..	78	7·4	19·3	4·5	·8	..	32·0
30–	12	25	6	5	3	51	5·0	10·2	2·5	2·1	1·2	20·9
35–	2	6	5	..	..	13	·8	2·5	2·1	..	..	5·3
40–	3	..	2	1	..	6	1·2	..	·8	·4	..	2·5
45–	..	..	2	3	1	6	..	..	·8	1·2	·4	2·4
50–	..	1	..	1	..	2	..	·4	..	·4	..	·8
55	1	..	..	..	..	1	·4	..	..	..	..	·4
	62	132	33	13	4	244	25·4	54·1	13·6	5·3	1·6	100·

Bombay Civilians.

The first report of Mr. Davies on the Bombay Civil Service Fund is dated as far back as 20th February, 1836. He collected the experience of the fund for 29 years, from its commencement to May, 1833, on the assumption that all the members were 20 years of age on their arrival in India, and then compared the results with Mr. Prinsep's table of the Bengal Civil Service from 1790 to 1831, as given in the first volume of the "Journal of the Asiatic Society." His own facts are but few in number, but they show a remarkable uniformity at all ages under 50, fluctuating between 2·35 and 2·60 per cent., and at the younger ages considerably exceeding the rates of mortality in Bengal. The following is the summary:—

Experience of the Bombay Civil Fund, Twenty-Nine Years, to 183 .

Ages.	Exposed to Risk.	Died.	Mortality per Cent.
20–	954	25	2·60
25–	724	17	2·35
30–	570	15	2·63
35–	410	10	2·44
40–	294	7	2·38
45–	193	5	2·59
50–	110	5	4·57
55–	23	1	4·26
60–62	8	1	12·50
	3,287	86	2·62

From a paper which was furnished by two of the East India Directors, Messrs. Ravenshaw and Loch, comprising a summary of the years 1805 to 1822, for the civil services in Bengal, Madras, and Bombay, it appeared that there were, on an average of each year, 431 living in Bengal, 226 in Madras, and 103 in Bombay, and the rates per cent. of deaths during the period were respectively 2·34, 2·30, and 3·08 per cent.

The experience of the Bombay Civil Fund bore out the observation generally made, that married life is subject to less mortality than single life, the rate of the former being only 2·51 per cent., compared with 2·62 per cent., above given.

As to retired members, he proposed to recommend the use of the Northampton Table at advanced ages, as allowing for the deterioration of health in Indian lives; though the experience of the fund really showed only 9 deaths above 47, whilst 10⅞ might have been expected by the use of that table. The table he used for the valuation, was constructed from the actual experience before 47, and from that age continued by the Northampton Table. By this table for a constant community of 170 persons living, at ages 20 to 45, 9·20 would have to be sent out annually, 4·86 would retire, and 4·34 would die.

Mr. Neison, in his report on this fund, 24th June, 1861, after referring to his other reports for information on the rate of mortality amongst European lives in India, copies word for word his own observations, tables, and comparisons in his report on the Bengal Civil Service Fund in 1852, and finally adopts the same table as he there gives, both for the active and retired services, starting only with the number 79,792, instead of 100,000, as living at 20.

It does not appear, therefore, that we have any original data from the experience of the Bombay Civil Fund since the small table furnished by Mr. Davies in 1836.

Madras Civilians.

In the first report of Mr. Davies, dated 9th March, 1850, on the Madras Civil Fund, he seems to have been unable to obtain the ages of the retired members or of their wives, or the numbers and ages of their children. He appears to have used the tables in his report of the Madras Military Fund for the valuation of the pensions, till death or marriage of the widows and daughters; and the single life table for females; and the joint life table for husband and wife, from the tables of the Bombay Civil Fund, in his report of 1836. No original data are here obtained.

Following this was Mr. Neison's report, dated 27th December, 1852, in which he repeats word for word his remarks on mortality which appear in his report for the Bengal Civil Service, dated a few days earlier, namely, on the 14th December in the same year. He, in conclusion, uses the same table both for active and retired service which we have before described, going back, however, to the basis of 100,000 as entering at age 20, and on which his subsequent monetary tables are computed.

But a subsequent report, bearing date the 20th July, 1855, furnishes some original data which are worth examining, relating to the mortality which was observed according to years of service, having the opportunity to compare with them a similar return which I have drawn up from the records of the Bengal Civil Service, from 1790 to 1842, amongst the members who were on service in India.

In the Bengal Civil Service the average age of arrival in India on the whole period, 1790 to 1842, was about 18½; but since 1820, it appears to have increased, and latterly may be taken as nearer age 20. By assuming the latter age for the commencement of observation, the rate of mortality in the annexed table will be found to correspond very nearly with the table under ages compiled under Mr. Prinsep's instructions, but after the first fifteen years is much higher than the rates observed in the most recent data from 1801 to 1858. There are scarcely any retirements under 25 years' service, then they increase rapidly up to 40–45, when they are upwards of 11 per cent. per annum, and the mortality diminishes in proportion.

In Mr. Neison's facts from the Madras Civil Fund, it will be

noticed, from the very long periods of service of some of the members, that those who have retired up to 1st January, 1854, are included under observation. The mortality, therefore, should only be compared for the first 25 years of service, and it will be found generally in Madras to be about 70 per cent. of that in Bengal.

Years of Service.	Bengal Civil Service, 1790–1842.				Madras Civil Fund, 1792–1854.	
	Exposed to Risk.	Rates per Cent. Died.	Resigned.	Retired.	Exposed to Risk.	Per Cent. Died.
0–	4,110	2·16	·51	..	3320·	1·39
5–	4,178	2·08	·79	..	2863·5	1·64
10–	3,366	1·37	·59	..	2444·	1·31
15–	2,338	2·61	·98	·04	2052·	1·85
20–	1,628	2·95	1·90	·06	1675·5	1·91
25–	944	3·49	·95	5·82	1326·5	1·81
30–	464	4·31	1·51	7·54	1069·	2·53
35–	183	5·46	..	10·38	836·	2·75
40–	72	2·78	2·78	11·12	613·5	3·26
45–	18	11·11	..	..	425·	2·59
50–	4	..	..	..	285·5	4·55
55–	..	..	..	..	150·5	8·64
60–	..	..	..	..	39·5	17·72
65–	..	..	..	..	7·	14·29
70–75	..	..	..	..	1·	100·
	17,305	2·30	·84	·69	17108·5	1·96
Total number	..	398	146	119	..	335

In reference to the mortality amongst civilians in India, the general conclusions at which we arrive, are—

1. That a considerable diminution has taken place of late years in the mortality at the middle ages, 20 to 35, and at all ages, if we compare it with the earlier observations of the present century.

2. That a very marked distinction may be observed in favour of married life.

3. That as compared with Farr's English healthy life table, the difference varies from ½ to 1 per cent. higher between the ages 20 and 55, after which it fluctuates, but is generally scarcely higher than the English rates.

In reference to the rate of marriage—

1. That the rate of marriage amongst bachelors is much higher at every age than in the peerage of Great Britain, and though at

ages under 30, it may be about 25 per cent. less than that of the general population, yet at all other ages it is considerably more.

2. That marriages take place at a much earlier period than in the military service, and on the average of all ages under 40, the rate is nearly double.

3. The same remark applies to widowers, whose marriage-rate under the age of 45 is considerably higher amongst the civil than the military service, though not more than 70 per cent. of that of the general population of England and Wales.

I trust that the few statistics here recorded may lead to a more careful collection in the books of the Indian Annuity and Pension Funds, from which so much information on the families of members can be readily obtained. They may throw light not merely on the relative mortality of India and this country—both subjects at the present time of the highest interest—but to the elucidation of many novel questions, which an accurate register of family statistics could not fail to afford us.

This subject may be further illustrated by some facts which have been collected recently in an interesting paper read before the Statistical Society by Mr. P. M. Tait, and published in the *Statistical Journal.* The Eurasians, as the name indicates, are a mixed race, the descendants of European (originally to a great extent Portuguese) and Asiatic parents. Latterly, the British is the predominant European element; but the name appears applied indiscriminately to the children of other colonists—Jews, Syrians, Christian Arabs, Armenians, Persians, Affghans, Danes at Serampore, Chinese and Americans. They are looked upon with some prejudice by the natives, being described as having the vices of the natives and Europeans, without the probity of the latter; but they are much employed in the inferior Government offices, and some Indian officers under whom they have served bear witness to their quickness at computation, intelligence, probity and unquestioned loyalty. They form a large proportion of the members of the Uncovenanted Service Pension Fund. Out of 945 who entered in 20 years ending 30th April, 1857, there were 693 of this class or 73 per cent., 244 Europeans or 26 per cent., and the remaining 8, or about 1 per cent., were undescribed. The dates of birth, entry, death, or withdrawal, were all verified for the 20 years; and comparing the results of the whole Fund, of the Eurasians only, and of the Bengal Civil Service from 1801 to 1858, which we have already examined, the following short table comprises the principal facts:—

Ages.	Uncovenanted Service Pension Fund, 20 Years to 30th April, 1857.						Bengal Civil Service, 1801 to 1857 inclusive. Mortality per Cent.
	Total Members.			Eurasians only.			
	Exposed to Risk.	Died.	Rate per Cent.	Exposed to Risk.	Died.	Rate per Cent.	
21–25	186·	..	..	173·5	..	..	1·65
26–30	747·	12	1·61	624·	7	1·12	2·02
31–35	1275·5	17	1·33	1019·	13	1·28	1·47
36–40	1329·5	25	1·88	1023·5	19	1·86	2·16
41–45	998·5	35	3·51	749·	28	3·74	2·12
46–50	683·	19	2·78	448·5	11	2·45	2·14
51–55	421·	24	5·71	283·5	16	5·64	4·29
56–60	238·5	7	2·94	147·5	4	2·71	3·77
61–65	115·5	15	12·99	76·	9	11·84	18·18
66–70	52·5	5	9·52	33·5	4	11·94	66·67
71–75	9·	3	33·33	3·	2	66·67	..
76–79	4·	1	25·	..	..	..	..
	6060·0	163	2·69	4581·	113	2·47	1·94

Hitherto the mortality of Eurasians has been thought to be considerably greater than that of Europeans, and some Assurance Companies declined them at European rates of premium; but at ages under 40 it seems that about 1·35 Eurasians die per cent., and 1·76 of European civilians. At older ages the reverse is shown, as above. It is probable, however, that with the recent improvement in European life in India, the difference would be found scarcely perceptible, even at the younger and most exposed ages.

If space permitted me to make a full comparison with the mortality of natives of India—soldiers and civilians—we should have to consult the admirable reports with which Colonel Sykes has from time to time for more than twenty years enriched the pages of the *Statistical Journal.*

I could not, however, conclude this part of the subject without a brief allusion to the recent and very elaborate "Report of the Commissioners appointed to inquire into the Sanitary State of the Army in India," in which our distinguished President of this Section, Dr. Farr, took so conspicuous a part. The fullest evidence was taken upon every subject that affects the health or mortality of the Indian army, the causes of the excess of the death-rate amongst Europeans as compared with natives, and the remedies suggested for the almost entire disappearance of such excess. The recommendations will be principally effective in bettering the condition of the common soldier; but some of them, such as the selection of hill stations, the improvement of barracks, &c., would no doubt

TABLE A (*Bengal Civil Service*).—*Experience of Mortality in the* 58 *Years*, 1801 *to* 1858 *inclusive, in Quinquennial Periods of Age.*

Ages.	Active Service.							Retired Annuitants.				
	Entered.	Lived.	Died.	Retired as Annuitants.	Withdrew.	Living to 1859.	Col. (2) − (Col. 1, 3, 4, 5, 6)/2 Exposed to Risk.	Entered.	Lived.	Died.	Living to 1859.	Col. (2) − (Col. 1, 3, 4)/2 Exposed to Risk.
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(1)	(2)	(3)	(4)	(5)
14–	760	1,285	20	..	3	..	893·5		..	..	..	..
20–	454	5,233	81	..	18	119	4897·		..	..	..	..
25–	1·	4,417	87	..	27	100	4309·5		..	..	..	..
30–	..	3,490	50	1	14	99	3408·	1	2	..	..	1·5
35–	..	2,666	56	5	11	71	2594·5	5	16	1	1	12·5
40–	..	2,054	42	62	10	36	1979·	64	126	1	1	93·
45–	..	1,076	21	112	2	56	980·5	126	694	7	29	613·
50–	..	365	14	42	1	20	326·5	53	831	14	45	775·
55–	..	117	4	12	..	6	106·	23	744	18	33	707·
60–	..	28	4	7	..	1	22·	9	568	20	25	541·
65–	..	2	..	1	..	..	1·5	3	346	20	29	320·
70–	..	..	..	..	..	..	..	3	156	9	16	142·
75–80	..	..	..	..	..	..	..	..	52	9	9	43·
	1,215	20,733	379	242	86	508	19518·	287	3,535	99	188	3248·

TABLE B (*Bengal Civil Fund*).—*Experience for* 13 *Years*, 1850 *to* 1862 *inclusive, amongst Members who were either Bachelors, Married Men, or Widowers, on* 1*st Jan.*, 1850, *or who entered either of those Classes since.* (*Active and Retired combined.*)

Ages.	Bachelors.							Married Men.							Widowers.						
	Entered.	Lived.	Died.		With-drew.	Mar-ried.	Living, 1 Jan., 1863.	Entered.	Lived.	Died.		With-drew.	Became Widowers	Living, 1 Jan., 1863.	Entered.	Lived.	Died.		With-drew.	Re-married.	Living, 1 Jan., 1863.
	(1)	(2)	(3) Killed.		(4)	(5)	(6)	(1)	(2)	(3) Killed.		(4)	(5)	(6)	(1)	(2)	(3) Killed.		(4)	(5)	(6)
19–	471	1,354	5	8	5	87	1	102	201	..	..	1	3	19	3	5	..	..	..	..	1
25–	59	1,049	7	14	12	84	106	133	645	3	4	1	13	57	13	36	..	..	..	6	4
30–	11	407	1	2	3	51	88	87	791	4	8	2	7	55	10	36	..	2	..	4	2
35–	25	208	..	1	1	14	21	78	715	4	10	2	14	71	23	75	1	1	..	8	6
40–	22	226	..	2	5	6	19	75	715	2	4	1	4	43	6	61	1	1	..	6	3
45–	7	191	..	3	10	6	8	29	747	2	13	5	5	26	11	81	..	1	1	4	4
50–	2	96	1	3	3	2	2	14	505	..	11	1	3	66	7	72	..	..	..	3	3
55–	1	18	..	..	..	1	12	7	204	..	4	1	2	29	2	46	..	1	..	1	7
60–	..	5	..	..	..	..	3	2	93	..	2	.	2	16	2	10	..	2	..	..	2
65–	..	2	..	1	..	..	..	2	33	..	2	..	..	3	..	7	..	..	..	..	1
70–	1	4	..	..	..	..	..	..	17	..	..	..	..	2	..	5	..	..	..	..	1
75–	..	5	..	1	..	..	..	..	10	..	1										
80–	..	..	..	..	..	..	..	..	1	..	1										
	599	3,565	14	35	39	251	260	529	4,677	15	60	14	53	387	77	434	2	8	1	32	34

In Table A the ages are those last birthday, but in Table B they are deduced from the year of birth to the year of event, so that the number exposed to risk would be Col. (2)$-\frac{\text{Col. } 1,\ 3,\ 4,\ 5,}{2}$ and the ages observed would be $x-\frac{1}{2}$ to $x+\frac{1}{2}$. The average age of retirement may be taken at about 48.

incidentally benefit the European officers also. In the report and appendix the summaries of the facts relating to Europeans in the civil or military services are compared. The general tenor of the report leads irresistibly to the conclusion that the great mortality, which formerly decimated the Indian armies, might, by judicious arrangements, be reduced to the ordinary rate amongst European civilians there; whilst the mortality amongst the latter has for many years undergone so great an improvement, as to present at some ages no very striking contrast with that of similar classes in this country; a remarkable proof that the science of statistics is not (as it used to be thought) a mere dry and tedious marshalling of figures, but an eminently practical and useful study, leading, even in the small part of its domain which we are now exploring, to suggestions which may be the means of preserving thousands of lives, and substituting the enjoyments of healthy existence for the uncontrolled ravages of disease and death.

A Budget of Paradoxes. *By* Professor De Morgan.

No. XIV. 1830—1833.

(Continued from p. 232.)

Demonville.—A Frenchman's Christian name is his own secret, unless there be two of the surname. M. Demonville is a very good instance of the difference between a French and English discoverer. In England there is a public to listen to discoveries in mathematical subjects made without mathematics: a public which will hear, and wonder, and think it possible that the pretensions of the discoverer have some foundation. The unnoticed man may possibly be right: and the old country-town reputation which I once heard of, attaching to a man who "had written a book about the signs of the zodiac which all the philosophers in London could not answer," is fame as far it goes. Accordingly, we have plenty of discoverers, who, even in astronomy, pronounce the learned in error because of mathematics. In France, beyond the sphere of influence of the Academy of Sciences, there is no one to cast a thought upon the matter: all who take the least interest repose entire faith in the Institute. Hence the French discoverer turns all his thoughts to the Institute, and looks for his only hearing in that quarter. He therefore throws no slur upon the means of knowledge, but would say, with M. Demonville—"A l'égard de M. Poisson, j'envie loyalement la millième partie de ses connaissances mathématiques,

pour prouver mon système d'astronomie aux plus incrédules." This system is that the only bodies of our system are the earth, the sun, and the moon; all the others being illusions, caused by reflexion of the sun and moon from the ice of the polar regions. In mathematics, addition and subtraction are for men; multiplication and division, which are in truth creation and destruction, are prerogatives of Deity. But *nothing* multiplied by *nothing* is *one*. M. Demonville obtained an introduction to William the Fourth, who desired the opinion of the Royal Society upon his system: the answer was very brief. The King was quite right; so was the Society: the fault lay with those who advised His Majesty on a matter they knew nothing about. The writings of M. Demonville in my possession are as follows. The dates—which were only on covers torn off in binding—were about 1831–34:—

"Petit cours d'astronomie"; followed by "Sur l'unité mathématique." —Principes de la physique de la création implicitement admis dans la notice sur le tonnerre par M. Arrago.—Question de longitude sur mer.—Vrai système du monde (pp. 92). Same title, four pages, small type. Same title, four pages, addressed to the British Association. Same title, four pages, addressed to M. Matthieu. Same title, four pages, on M. Bouvard's report.—Résumé de la physique de la création; troisième partie du vrai système du monde.

The quadrature of the circle discovered, by Arthur Parsey, author of the "Art of miniature painting." Submitted to the consideration of the Royal Society, on whose protection the author humbly throws himself. London, 1832, 8vo.

Mr. Parsey was an artist, who also made himself conspicuous by a new view of perspective. Seeing that the sides of a tower, for instance, would appear to meet in a point if the tower were high enough, he thought that these sides ought to slope to one another in the picture. On this theory he published a small work, of which I have not the title, with a Grecian temple in the frontispiece, stated, if I remember rightly, to be the first picture which had ever been drawn in true perspective. Of course the building looked very Egyptian, with its sloping sides. The answer to his notion is easy enough. In July, 1831, reading an article on squaring the circle, and finding that there was a difficulty, he set to work, got a light denied to all the mathematicians in—some would say through—a crack, and advertised in the *Times* that he had done the trick. He then prepared this work, in which, those who read it will see how, he showed that 3·14159.... should be 3·0625. He might have found out his error by *stepping* a draughtsman's circle with the compasses.

Perspective has not had many paradoxes. The only other one I remember is that of a writer on perspective, whose name I forget, and whose four pages I do not possess. He circulated remarks on my notes on the subject, published in the *Athenæum*, in which he denies that the stereographic projection is a case of perspective, the reason being that the whole hemisphere makes too large a picture for the eye conveniently to grasp at once. That is to say, it is no perspective because there is too much perspective.

Principles of geometry familiarly illustrated. By the Rev. W. Ritchie, LL.D. London, 1833, 12mo.

A new exposition of the system of Euclid's Elements, being an attempt to establish his work on a different basis. By Alfred Day, LL.D. London, 1839, 12mo.

These works belong to a small class which have the peculiarity of insisting that in the general propositions of geometry a proposition gives its converse; that "Every B is A" follows from "Every A is B." Dr. Ritchie says, "If it be proved that the equality of two of the angles of a triangle depends *essentially* upon the equality of the opposite sides, it follows that the equality of the opposite sides depends *essentially* on the equality of the angles." Dr. Day puts it as follows:—

"That the converses of Euclid, so called, where no particular limitation is specified or implied in the leading proposition, more than in the converse, must be necessarily true; for as by the nature of the reasoning the leading proposition must be universally true, should the converse not be so, it cannot be so universally, but has at least all the exceptions conveyed in the leading proposition, and the case is therefore unadapted to geometric reasoning, or, what is the same thing, by the very nature of geometric reasoning, the particular exceptions to the extended converse must be identical with some one or other of the cases under the universal affirmative proposition with which we set forth, which is absurd."

On this I cannot help transferring to my reader the words of the Pacha when he orders the bastinado—"May it do you good!" A rational study of logic is much wanted to show many mathematicians, of all degrees of proficiency, that there is nothing in the *reasoning* of mathematics which differs from other reasoning. Dr. Day repeated his argument in *A Treatise on Proportion*, London, 1840, 8vo. Dr. Ritchie was a very clear-headed man. He published, in 1818, a work on arithmetic, with rational explanations. This was too early for such an improvement, and nearly the whole of this excellent work was sold as waste paper. His elementary introduction to the Differential Calculus was drawn up while he was learning the subject late in life. Books of this sort are often very effective on points of difficulty.

Letter to the Royal Astronomical Society in refutation of mistaken notions held in common, by the Society, and by all the Newtonian philosophers. By Capt. Forman, R.N. Shepton-Mallet, 1833, 8vo.

Capt. Forman wrote against the whole system of gravitation, and got no notice. He then wrote to Lord Brougham, Sir J. Herschel, and others I suppose, desiring them to procure notice of his books in the reviews: this not being acceded to, he wrote (in print) to Lord John Russell to complain of their "dishonest" conduct. He then sent a manuscript letter to the Astronomical Society, inviting controversy; he was answered by a recommendation to study dynamics. The above pamphlet was the consequence, in which, calling the Council of the Society "craven dunghill cocks," he set them right about their doctrines. From all I can learn, the life of a worthy man and a creditable officer was completely embittered by his want of power to see that no person is bound in reason to enter into controversy with every one who chooses to invite him to the field. This mistake is not peculiar to philosophers, whether of orthodoxy or paradoxy; a majority of educated persons imply, by their modes of proceeding, that no one has a right to any opinion which he is not prepared to defend against all comers.

David and Goliath, or an attempt to prove that the Newtonian system of astronomy is directly opposed to the Scriptures. By Wm. Lauder, Sen., Mere, Wilts. Mere, 1833, 12mo.

Newton is Goliath; Mr. Lauder is David. David took five pebbles; Mr. Lauder takes five arguments. He expects opposition; for Paul and Jesus both met with it.

No. XV. 1834—1835.

A Treatise on the Divine System of the Universe, by Captain Woodley, R.N., and as demonstrated by his universal time-piece, and universal method of determining a ship's longitude by the apparent true place of the moon; with an introduction refuting the solar system of Copernicus, the Newtonian philosophy, and mathematics. 1834, 8vo.
Description of the universal time-piece. (4 pp. 12mo.)

I think this divine system was published several years before, and was republished with an introduction in 1834. Capt. Woodley was very sure that the earth does not move; he pointed out to me, in a conversation I had with him, something—I forget what—in the motion of the Great Bear, visible to any eye, which could not possibly be if the earth moved. He was exceedingly ignorant, as

the following quotation from his account of the usual opinion will show:—

"The north pole of the Earth's axis deserts, they say, the north star or pole of the Heavens, at the rate of 1° in 71⅗ years.... The fact is, nothing can be more certain than that the Stars have not changed their latitudes or declinations *one degree* in the last 71⅗ years."

This is a strong specimen of a class of men by whom all accessible persons who have made any name in science are hunted. It is a pity that they cannot be admitted into scientific Societies, and allowed fairly to state their cases, and stand quiet cross-examination, being kept in their answers very close to the questions, and the answers written down. I am perfectly satisfied that if one meeting in the year were devoted to the hearing of those who chose to come forward on such conditions, much good would be done. But I strongly suspect few would come forward at first, and none in a little while: and I have had some experience of the method I recommend, privately tried. Capt. Woodley was proposed, a little after 1834, as a Fellow of the Astronomical Society; and, not caring whether he moved the sun, or the earth, or both—I could not have stood *neither*—I signed the proposal. There was so little feeling against his opinions, that he only failed by a fraction of a ball. Had I myself voted, he would have been elected; but being engaged in conversation, and not having heard the slightest objection to him, I did not think it worth while to cross the room for the purpose. I regretted this at the time, but had I known how ignorant he was I should not have supported him. Probably those who voted against him knew more of his books than I then did.

I remember no other instance of exclusion from a scientific Society on the ground of opinion, even if this be one; of which it may be that ignorance had more to do with it than paradoxy. Mr. Frend, a strong anti-Newtonian, was a Fellow of the Astronomical Society, and for some years in the Council. Lieutenant Kerigan was elected to the Royal Society at a time when his proposers must have known that his immediate object was to put F.R.S. on the title-page of a work against the tides. To give all I know, I may add that the editor of some very ignorant bombast about the "forehead of the solar sky," who did not know the difference between *Bailly* and *Baily*, received hints which induced him to withdraw his proposal for election into the Astronomical Society. But this was an act of kindness; for if he had seen Mr. Baily in the chair, with his head on, he might have been political historian enough to faint away.

De la formation des Corps. Par Paul Laurent. Nancy, 1834, 8vo.

Atoms, and ether, and ovules or eggs, which are planets, and their eggs, which are satellites. These speculators can create worlds, in which they cannot be refuted: but none of them dare attack the problem of a grain of wheat, and its passage from a seed to a plant, bearing scores of seeds like what it was itself.

An account of the Rev. John Flamstead, the First Astronomer-Royal... By Francis Baily, Esq. London, 1835, 4to. Supplement, London, 1837, 4to.

My friend Francis Baily was a paradoxer: he brought forward things counter to universal opinion. That Newton was impeccable in every point was the national creed; and failings of temper and conduct would have been utterly disbelieved, if the paradox had not come supported by very unusual evidence. Anybody who impeached Newton on existing evidence might as well have been squaring the circle, for any attention he would have got. About this book I will tell a story. It was published by the Admiralty for distribution; and the distribution was intrusted to Mr. Baily. On the eve of its appearance, rumours of its extraordinary revelations got about, and persons of influence applied to the Admiralty for copies. The Lords were in a difficulty: but on looking at the list they saw names, as they thought, which were so obscure that they had a right to assume Mr. Baily had included persons who had no claim to such a compliment as presentation from the Admiralty. The Secretary requested Mr. Baily to call upon him. "Mr. Baily, my Lords are inclined to think that some of the persons in this list are perhaps not of that note which would justify their Lordships in presenting this work."—"To whom does your observation apply, Mr. Secretary?"—"Well now, let us examine the list; let me see; now,—now,—now,—come!—here's Gauss—*who's Gauss?*"—"Gauss, Mr. Secretary, is the oldest mathematician now living, and is generally thought to be the greatest." —"O-o-oh! Well, Mr. Baily, we will see about it, and I will write you a letter." The letter expressed their Lordships' perfect satisfaction with the list.

God's creation of the Universe as it is, in support of the Scriptures. By Mr. Finleyson. Sixth Edition, 1835, 8vo.

This writer, by his own account, succeeded in delivering the famous Lieut. Richard Brothers from the lunatic asylum, and tending him, not as a keeper but as a disciple, till he died. Brothers was, by his own account, the nephew of the Almighty, and Finleyson ought to have been the nephew of Brothers. For

Napoleon came to him in a vision, with a broken sword and an arrow in his side, beseeching help: Finleyson pulled out the arrow, but refused to give a new sword; whereby poor Napoleon, though he got off with life, lost the battle of Waterloo. This story was written to the Duke of Wellington, ending with "I pulled out the arrow, but left the broken sword. Your Grace can supply the rest, and what followed is amply recorded in history." The book contains a long account of applications to Government to do three things: to pay £2,000 for care taken of Brothers, to pay £10,000 for discovery of the longitude, and to prohibit the teaching of the Newtonian system, which makes God a liar. The successive administrations were threatened that they would have to turn out if they refused, which, it is remarked, came to pass in every case. I have heard of a joke of Lord Macaulay, that the House of Commons must be the Beast of the Revelations, since 658 members, with the officers necessary for the action of the House, makes 666. Macaulay read most things, and the greater part of the rest: so that he might be suspected of having appropriated as a joke one of Finleyson's serious points—"I wrote Earl Grey upon the 13th of July, 1831, informing him that his Reform Bill could not be carried, as it reduced the members below the present amount of 658, which, with the eight principal clerks or officers of the House, make the number 666." But a witness has informed me that Macaulay's joke was made in his hearing a great many years before the Reform Bill was proposed; in fact, when both were students at Cambridge. Earl Grey was, according to Finleyson, a descendant of Uriah the Hittite. For a specimen of Lieut. Brothers, this book would be worth picking up. Perhaps a specimen of the Lieutenant's poetry may be acceptable: Brothers *loquitur*, remember:—

"Jerusalem! Jerusalem! shall be built again!
  More rich, more grand than ever;
And through it shall Jordan flow! (!)
  My people's favourite river.
There I'll erect a splendid throne,
  And build on the wasted place;
To fulfill my ancient covenant
  To King David and his race.
* * * *
Euphrates' stream shall flow with ships,
  And also my wedded Nile!
And on my coast shall cities rise,
  Each one distant but a mile.
* * * *

My friends the Russians on the north,
  With Persees and Arabs round,
Do show the limits of my land,
  Here! Here! then I mark the ground."

(*To be continued.*)

CORRESPONDENCE.

SOLUTION OF PROBLEM PROPOSED BY JUVENIS.

To the Editor of the Assurance Magazine.

SIR,—Referring to the question proposed by Juvenis, in the April Number of the *Magazine*, I send the particulars of the method which I adopted to find an approximate solution.

The question is, if I understand it correctly, "What is the value of a perpetuity to be enjoyed by 48, in the event of 55, 53, 51, and 50, all dying before him; the first payment to be made at the end of the year in which the *last* of these four lives should fail?"

The value of this perpetuity, at 4 per cent, will be equal to an assurance of £26, payable if 55, 53, 51, and 50, should all die before 48. In order to simplify the solution, I substituted a single life, corresponding to the value of an annuity on the *longest* of the four lives, and found the value of an assurance payable if the single life, thus obtained, should die before 48. The value was, at 4 per cent. and using the Carlisle tables, 3·7752 years' purchase of the rental of the estate.

There are, however, many things which would have an important bearing on this value; for instance, the state of 48's health and of that of the other lives, whether they are related to each other by blood, &c.; in short, actuaries would, in stating a value, be guided more by their own judgment than by any tabular or mathematical value, which can only, without great trouble, be approximate. If 48 were selling his interest, it is quite probable that he would only realise two years' purchase for it.

I am, Sir,
Yours obediently,
T. M.

THE SAME SUBJECT.

To the Editor of the Assurance Magazine.

SIR,—The suggestion made by Juvenis in the last April Number of our *Journal* is one which I, and I believe many others engaged in our pursuits, would be glad to see extensively acted upon.

The estimates of actuaries are now constantly made the subjects of discussion in courts of law and equity and before other tribunals; and the great difference of opinion which they so frequently exhibit becomes conspicuous, and tends to bring such estimates into doubt and general discredit. Any process, therefore, which will serve to bring this want of agreement

within reasonable limits is most desirable; and since the one suggested by Juvenis is obviously calculated to effect this object, it appears to me well deserving of our support, and I for one shall hope to see a constant succession of questions or cases put forward, and some agreement arrived at as to the principles in accordance with which they should be solved; for it is in the difference of the principles adopted for the solution, and not in the mere calculation, that the discrepancies most commonly arise—that is to say, the discrepancies are of a logical rather than of a mathematical kind. This will, I dare say, appear in the solutions to the question proposed by Juvenis, supposing more than one solution to be given.

It is not unlikely that the proposer contemplates a strictly mathematical solution, involving a minute investigation of the probabilities of survivorship amongst the lives he has enumerated. That would not be the view taken by an actuary before whom the case came in the ordinary way for an opinion. He would seek to discover what sum could be safely invested in the purchase of such a reversion, and what it would cost to assure against the contingencies affecting it; and he would find in that point of view that such a reversion was worthless. Hence we see that in this case, as in almost all others, more than one solution can be given; and our attention is directed to the importance of ascertaining what interpretations can be put upon a question, and of guarding against a solution intended for one of them being mistaken for another.

I am, Sir,
Your obedient servant,
A FELLOW OF THE INSTITUTE.

ON THE VALUE OF OPTIONS.

To the Editor of the Assurance Magazine.

Dear Sir,—In the last Number of your *Journal* a letter by Mr. Makeham is inserted, in which the writer endeavours to prove that the method which I had previously given for finding the single premium for a deferred annuity, with the condition that the premium shall be returnable (without interest) at death or, at the option of the purchaser, at any time before the annuity becomes payable, is defective, inasmuch as in his opinion it provides only for the deferred annuity and the return of the premium in the event of death; which return he assumes that I have made payable, with one year's interest thereon, at the end of the year in which the life fails.

My present object is, in the first place, to show that Mr. Makeham has entirely overlooked the very point on which alone the interest in the problem may be supposed to rest, as he "naively," but erroneously, remarks, "*that in assurances of this description the value of the policy always exceeds the premium paid upon it—a circumstance which does not depend upon the mode of computing the premium, but arises from the nature of the contingency itself*"; and in the second place I wish to point out, that in the solution which I have given the sum returnable is P_x, and is made *at the time of*, or *prior to*, death, and is not $P_x(1+i)$; nor is it made at the end of the year in which the life fails, as Mr. Makeham has assumed.

It can hardly be necessary for me to refute the assertion that "*the*

value of the policy always exceeds the premium paid upon it," as it must be obvious to all your readers that when, from the failing health of a person who has purchased an ordinary deferred annuity on his life, death is presumably certain to take place within the term for which the annuity was originally deferred, the value of such an assurance becomes absolutely *nothing;* in fact, if this were not so, the premium usually charged in such cases would be utterly inadequate to provide for the benefit secured.

Now it is just at this point, and under these circumstances, that, in the solution which I have given, it is assumed that the "option" will be exercised. In short, it is assumed that it will be exercised whenever (were there no option) the value of the policy would be *less* than the premium paid upon it, and then only. But this would take place not only when there was a high probability that the assured would die within the term of n years, but it would also obtain whenever it became evident that, although the annuitant might survive the term, the value of the annuity about to be entered upon would, from his then shattered health, be *less* than P_x, at which price nevertheless the value of such an assurance would, under the optional clause, be "arbitrarily fixed," notwithstanding Mr. Makeham's statement that the assured "*would be entitled in addition to an allowance from the Office for the surrender of the deferred annuity secured by the annual interest.*"

Of course the value in the former instance would not, as a rule, be very much less than P_x, but it is easy to conceive a case, in which it might be morally certain that a person would survive the term but yet not live to receive even the first year's payment of the annuity, when the value of the policy would fall to zero. In dealing with this problem the only difficulty which has ever been felt has been in assigning a value to the probability of a person requiring the repayment of his premium in anticipation of death, or when, from his general state of health, he would be unable to pass the usual medical examination which is necessary before surrendering an ordinary deferred annuity policy.

The formula $\frac{N_{x+n}}{(N_{x-1}-N_{x+n-1})(1-v)+D_{x+n}}$, given by Mr. Makeham in your last Number, provides only for the deferred annuity or the return of P_x *at the end of the year in which the life fails,* whilst the formula which I have given, $\left(\frac{N_{x+n}}{(N_x-N_{x+n})i+D_{x+n}}\right)$, will provide for the annuity or the return of P_x *at the moment of death* or *at any time previously,* whenever, from failing health or other circumstances, such a course may be deemed desirable; the latter being greater than the former, proves that the option has a *positive* value—a *negative* value is impossible.

Of the very small difference which the option made I was fully aware before Mr. Makeham's letter appeared; but as my primary object was to show the sufficiency of the premiums given by me as contrasted with those charged by the Government, I avoided trespassing on your valuable space, further than appeared necessary at the time, into questions which did not tend to elucidate that point.

I am, dear Sir,
Yours truly,

London, 25th August, 1865. J. W. STEPHENSON.

ON THE RATE OF MORTALITY AMONG SELECT LIVES.

To the Editor of the Assurance Magazine.

Sir,—Referring to the Note appended by you to Mr. Berridge's paper in last Number, in which you have been good enough to call attention to a paper of mine, I readily allow that selection as ordinarily practised does not eliminate all unsoundness; but I am not prepared to admit that if the value of selection at all ages were thoroughly tested and proved, it would establish the accuracy of the theory that in the absence of unsoundness the declension in vitality from early youth (if not from birth) till death is in a progression uniformly accelerated.

I now consider that it may be held as settled that there is no reliable evidence upon which to maintain that there is a greater mortality for one year in a select life from about 40 to 45 than in one from about 20 to 25; and that, on the contrary, the evidence before us would lead rather to establish the doctrine that there is no materially greater risk in the assurance of the elder lives for a year.

I think it quite likely that what you designate anomalies in the Carlisle table are really the result of defective and limited observations; and, as a *statistical result*, the doctrine of the proportion of deaths, among healthy and sick living, increasing from early youth, may be quite true; but I do not think it is likely to be true as a physiological law of the individual or of healthy lives. At all events the general table can have no legitimate bearing on such a question; for I do not think it admits of any question that the mortality for the first year among say 10,000 lives of the ordinary population, of ages similar to those generally assured, is more than twice, and probably about three times, the actual mortality if the lives were properly selected.

I trust that in the greater attention which is now paid to "Experience" by Offices it will be made a point, if not to exhibit the mortality during every year of the duration of policies, at all events to show it during the first year. I feel confident that more real additional knowledge would be obtained by an extensive observation of this limited nature than by a large general Experience over all years. I think we know pretty well how the general Experience stands, but the bearing upon the laws of mortality of the undoubted enormous difference of the mortality among select lives for a year, as compared with that of the general population, has been, comparatively speaking, almost left uninvestigated.

I hope theories may soon give place to ascertained laws, but in the meantime the theory of the mortality among select lives which approves itself most to my mind is, that (while selectness continues) from birth to the prime of manhood—I would not fix the age, but say 35—the mortality diminishes by a lessening ratio, and that after the prime of manhood it increases by an accelerated ratio. All this may be quite consistent with the statistics of the mortality of the general mass of good and bad lives.

I am, Sir,
Yours faithfully,
WILLIAM SPENS.

Glasgow, August 31, 1865.

THE

ASSURANCE MAGAZINE,

AND

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the Principles to be observed in the Construction of Mortality Tables. *By* WILLIAM MATTHEW MAKEHAM, *Fellow of the Institute of Actuaries.*

[Read before the Institute, 27th November, 1865.]

THE remarks which I have to offer for the consideration of this meeting have reference not to the deduction of the probabilities of living and dying from the facts observed, but to the mode of dealing with those probabilities, in their rough state, with the view of rendering them fit for the purpose for which they may ultimately be required.

In no observations which have hitherto been made by the exact enumeration of the living and dying at each separate year of age, have the facts observed been sufficiently numerous to render the process of adjustment unnecessary—or, at least, undesirable. Few, I think, will be disposed to contend against the propriety of adopting *some* such process; although, as might be expected, the extent to which the alteration of the original figures should be carried, is a question upon which considerable difference of opinion undoubtedly exists. The great importance of the subject will, I trust, be a sufficient apology for the following attempt to bring about some approach to an agreement upon this question.

Now I submit, with all deference, that one cause of the want

of agreement referred to may possibly arise from our not distinctly defining the precise object for which the resulting table is supposed to be required. If all we want to attain is a true statement of the rate of mortality at each age which *has* prevailed among the lives observed, it is clear that the alteration of a single figure of the actual results is inadmissible. Those figures, and those alone, give us the information we are supposed to be in search of.

But if, on the other hand, we wish to obtain, not merely the ratios of the actual number of deaths which have occurred at each age, but an idea of the law of mortality of which those figures are but the rough indications; in other words, if we wish to obtain an approximation to the ratios which would probably have resulted if the observations had been unlimited in point of number—a desideratum not only instructive in itself, but essential to a correct determination of the *probable future* experience; if this be our object, it will, I think, be admitted that a process of adjustment should be used which merely softens down the inequalities occurring at ages in immediate juxtaposition, without destroying any distinctly marked feature observable at particular periods of life, whether such feature may be supposed to belong to the normal law of mortality, to which mankind in general is subject, or whether it be considered peculiar to the particular class or community upon which the observations are based.

As an example of the kind of adjustment here referred to I may instance the observations made by the late Mr. Finlaison on the mortality of the male nominees of the Government tontines and life annuities. In observation 15 (which comprises the whole of the male lives), we find that from the age of 16 to the age of 23, the annual rate of mortality rises rapidly from about 8 to a maximum of 15 per 1,000; after which it gradually diminishes to a minimum, at age 34, of 11·7 per 1,000. This remarkable deviation from the general law of progression in the rate of mortality from age to age is not destroyed by the mode of adjustment adopted by Mr. Finlaison; and confirmed as it is by each of the separate classes of the same observations, and also by other similar but totally independent observations, there can be no question that it truly represents a distinctive feature in the mortality prevailing among the lives upon which these particular observations are based.

Such being the case, I think the conclusion is inevitable, that if our object be (as we have supposed) simply to ascertain, as nearly as possible, the true character of the law of mortality to which the community (taken in the aggregate) *has* been exposed,

and which will probably *in future* prevail among the same or a similar community; then, as before observed, a method of adjustment which (like Mr. Finlaison's) merely corrects the abrupt transitions at consecutive ages caused by the insufficiency of the data, and preserves any distinctly marked peculiarity in the results, is all that we are justified in applying. By doing more than this, the objection urged by Professor De Morgan, that the tables are thereby deprived of a portion of their value as the representations of physical *facts*, can neither be denied nor rebutted.

And here it might not unnaturally be supposed that this conclusion really involves the whole question in discussion; for, it may be asked, does not a table so adjusted form the true and proper standard for estimating the probabilities of life among the same or any similar community—the only practical object for which such tables are usually required?

To this I answer, undoubtedly it does, *provided only* (and this proviso is an important one) that nothing is known respecting the individual whose chances of life we are estimating beyond his age and the single fact that he is a member either of the particular community which has furnished the materials for the construction of the table, or of a class supposed to be subject to the same law of mortality.

For instance, we have seen that, according to the observations upon the Government male annuitants, the annual mortality at the age 23 is 15 per 1,000, while at an age 11 years greater it is 11·7 only. Now, if all we know of two particular individuals, of the respective ages 23 and 34, is that they are both included among the Government annuitants, or that they belong to the class of society from which those annuitants are principally recruited, then undoubtedly we shall rightly conclude that their relative chances of dying within a year are as 15 and 11·7; and if an Assurance Office were asked to undertake the risk upon the limited information supposed, the premium required for the younger life should exceed that required for the elder in the same proportion.

But if, on the other hand, besides the fact that the lives are Government annuitants, it is also known that both of them are at the time in sound health, of sober and temperate habits, and belong to a family free from any hereditary taint, does it *then* follow that the numbers in question will correctly denote the relative probabilities of death? That they will not truly represent the *absolute* magnitude of such probabilities is certain, and I think, for the reasons which I shall presently adduce, we shall not be far wrong

in supposing that they as little indicate their *relative* magnitude, or the ratio which the one risk bears to the other.

This brings us naturally to the third object which we may have in view in forming a table of mortality, viz., the construction of an instrument which shall serve as a measure of the value of *selected* life, that is, of the average duration of the lives of persons who, at the time when they form the subject of calculation, are known, or supposed, to be free from any exceptional cause of mortality.

In the first place, it requires but little experience to tell us that this object is of far greater general utility than either of the two which we have previously examined. It is very seldom indeed that pecuniary contracts are entered into, depending upon the contingencies of life, without *some* information being obtained respecting the state of health and the habits of the particular individual whose life is involved; and whether such information be favourable or unfavourable, it must, I think, be sufficiently obvious that a table constructed by the combination of the two classes can form but a very imperfect measure of the value of either.

It will, indeed, be readily perceived that the object which we are now considering can be perfectly attained only by an examination of the results of observations upon lives selected at each successive year of age, and the construction of a separate mortality table for each class. This subject did not escape the attention of the eminent actuaries by whom the mortality of the seventeen Assurance Offices was investigated; and I believe that in the examination which the Council of the Institute is now conducting into the present more extended experience among assured lives, the question is receiving that attention which its importance so justly demands. Mr. Higham's most able and interesting papers on this subject, read before the Institute on the 25th March, 1850, and the 31st March, 1851, are probably familiar to all present; and I think that, considering the important deductions exhibited by Mr. Higham, it must have occurred to most of us, that the observations upon the Government annuitants, valuable as they undoubtedly are, would have been rendered still more so, not only in a scientific point of view, but also as a necessary test of the adequacy of the prices charged by the Government in the grant of annuities at certain ages, if a complete investigation, conducted with the ability displayed in Mr. Higham's papers, had been made of their bearing upon this question.

Notwithstanding, however, the great interest and importance of

the subject of classification, I forbear, for two reasons, to pursue it further on this occasion. The first is, that there exist no data suitable for our purpose; for unfortunately, owing to a cause which I shall hereinafter refer to, the observations upon assured lives (important and useful as they are in other respects) do not, from their nature, admit of any correct deductions being drawn from them on this head; and secondly, I am of opinion that the complexity which would be introduced into the calculation of life contingencies, if, in each distinct class or community observed, a separate table of mortality were used for each different year of age, renders it extremely undesirable that any such practice should be adopted. The question which we have, then, to consider is, in what way we may obtain the best substitute for the perfect instrument which the circumstances of the case admit of.

Now, it appears to me that the only course open to us is, to adjust the tables which we propose to adopt for this purpose upon a somewhat broader basis than that described as applicable to the purpose previously referred to. In other words, I would first endeavour to ascertain, by an examination of the data best adapted for this object, what are the essential features of the natural or normal law of mortality, as exhibited in the effects produced by that gradual change in the vital powers of the human body which renders the chance of death so much greater to a healthy life of 70 or 80 than it is to a healthy life of 20 or 30. Having settled this question as satisfactorily as our limited means of observation will admit, we must then adjust or correct our table according to the model thus determined; and I am greatly mistaken if we shall not by these means obtain a truer standard for the value of life (having regard to the actual conditions of the case) than a more rigid adherence to the uncorrected results would have afforded us.

In order that the conclusions deduced from the investigation of the nature of the *normal* law of mortality may not be vitiated by possible errors in the collection of the data examined, I have considered it advisable to confine myself exclusively to tables formed upon exact enumerations of the numbers living and dying at each particular year of age. Whatever objections may be urged against some of the following observations in other respects, it must be admitted that in the essential point of arithmetical accuracy they are immeasurably superior to any observations derived from population returns and the public registers of deaths, however skilfully and judiciously such observations may be conducted.

The tables available for my purpose are limited (so far as my

knowledge extends) to nine, which, for reasons which will appear, I have arranged in the following order:—

(*Male Life.*)

1. Government Annuitants.*
2. Peerage Families.†
3. Assured Lives.‡
4. Members of Friendly Societies.§
5. Clergy of England and Wales.‖

(*Female Life.*)

6. Assured Lives.‡
7. Friendly Societies.§
8. Government Annuitants.*
9. Peerage Families.†

I.—*Observations on Male Life.—Annual Mortality per* 1,000.¶

Age.	Government Annuitants.	Peerage Families.	Assured Lives.	Friendly Societies.	Clergy.	Age.
18	10·9	7·4	8·5	6·7	..	18
23	15·1	11·6	8·4	7·3	..	23
28	13·4	10·1	9·1	7·6	4·6	28
33	11·8	7·1	9·3	8·0	6·4	33
38	14·0	10·5	11·3	9·5	7·1	38
43	14·0	11·0	13·1	11·1	9·2	43
48	14·9	13·9	17·1	13·6	11·3	48
53	23·2	16·4	22·4	17·3	18·5	53
58	29·2	19·7	32·3	25·2	22·9	58
63	40·8	33·2	40·6	29·8	34·5	63
68	61·7	51·0	53·7	48·1	49·8	68
73	81·7	78·5	78·9	66·6	83·0	73
78	114·3	104·3	120·2	102·6	125·9	78

* "Report of John Finlaison, Actuary of the National Debt, on the Evidence and Elementary Facts on which the Tables of Life Annuities are founded." Ordered by the House of Commons to be printed, 31st March, 1829.

† "On the Rate of Mortality prevailing amongst the Families of the Peerage during the 19th Century." By Arthur Hutcheson Bailey and Archibald Day, Esqs. (*Assurance Magazine*, vol. ix.)

‡ "Tables exhibiting the Law of Mortality deduced from the Combined Experience of Seventeen Life Assurance Offices;" 1843.

§ "Report and Tables on the Sickness and Mortality among the Members of Friendly Societies." (Alexander Glen Finlaison, Esq.) Ordered by the House of Commons to be printed, 16th August, 1853.

‖ "Observations in Reference to the Duration of Life amongst the Clergy in England and Wales." By the Rev. John Hodgson. With a Supplement by Samuel Brown, Esq. 1865.

¶ In this and the following table the annual rate of mortality at the age x is the average of the five years, $x-2$, $x-1$, x, $x+1$, and $x+2$; by which arrangement every item of the original observations has its due effect in the condensed tables here given. An exception, however, has been made in the case of the "Government Annuitants," which are taken, without alteration, from Mr. Finlaison's adjusted series.

II.—*Observations on Female Life.—Annual Mortality per* 1,000.

Age.	Assured Lives.	Friendly Societies.	Government Annuitants.	Peerage Families.	Age.
18	..	9·1	8·4	8·2	18
23	15·6	9·2	8·6	8·3	23
28	13·5	11·0	9·6	8·1	28
33	18·9	10·4	10·3	10·3	33
38	14·2	11·9	11·5	10·1	38
43	14·4	11·2	11·7	11·4	43
48	17·7	11·7	14·9	12·5	48
53	17·7	15·5	16·5	14·6	53
58	27·5	21·4	20·5	21·1	58
63	32·9	35·0	28·8	35·1	63
68	52·1	51·4	43·0	38·2	68
73	100·9	82·4	64·8	58·8	73
78	183·5	121·8	99·6	77·2	78

The one great and prominent characteristic common to these several observations—and, indeed, to all other observations on human mortality—is the gradual progression from age to age in the rate of mortality—the rate of increase, however, being in every instance much greater at the higher than the lower ages. This simple and obvious characteristic of the law of mortality was pointed out by Dr. Price, and indeed is consistent with the commonest notions of the nature of that law.

If, however, we examine more closely the several columns of observations upon *male* life, we find that this simple law of progression is, *in some of them*, subject to some disturbing influence, commencing apparently immediately after the age of 18 (at which our observations begin), and ceasing about the age of 38 or 40. We see, for instance, in the case of the "Government Annuitants" and the "Peerage Families" that the rate of mortality increases with considerable rapidity during the five years following the age of 18, and then gradually diminishes during the next ten years, after which it increases uninterruptedly during the remainder of the period observed. The same disturbing influence, however, is not perceptible in the other observations on male life.

If again we examine the observations upon *female* life, we find the whole of them free from this particular deviation from the general law of progression. In one of them, indeed (the assured lives), we see a diminution of the rate of mortality at age 28, but this is immediately followed by a much greater increase at age 33, after which another temporary decrease takes place. In the Friendly Societies, also, a decrease takes place twice, but this occurs at the ages 33 and 43. These fluctuations may possibly be owing

to some extent to the limited numbers upon which the observations are based; but I am inclined to suspect that the prevalence of speculative assurances upon the lives of females—to which some Offices, I am afraid, lend themselves too incautiously—may have something to do with it. At all events, as these peculiarities are confined to the observations of this particular kind, viz., the "Assured Lives" and "Friendly Societies," and further, as there is no uniformity in the deviations in the two observations, I think they may safely be discarded in an inquiry into the nature of the normal law of mortality.

Confining onr attention, therefore, to the abnormal feature observable in two of the observations on male life immediately after the age of 18—which, in fact, is the only one existing in different and independent observations—we have now to inquire to what cause it may probably be attributable. Now, we must bear in mind that the "annuitants," upon whose lives Mr. Finlaison's observations are founded, were differently circumstanced from the members of Assurance and Friendly Societies, who are, for the most part, brought together, from habits of prudence and foresight, for the purpose of making a provision for the future. The "annuitants," on the contrary, were, generally speaking (at the earlier ages), simply nominated by the person entitled to the annuity, having themselves no interest in the matter. They were, therefore, almost invariably chosen from among the families of the more affluent classes; to some extent, because such persons would not unreasonably be supposed to enjoy a greater average duration of life, but principally because they could be more easily traced in after life. They would, in fact, belong to a class of the community very similar to that of the peerage families, in which, as we have seen, the same remarkable deviation from the law of progression is observed. Now, when we consider the mode of life, which to many young men, not in *this* particular class only, but in *all* classes of the community, at about the ages we are referring to, appears to offer such attractions—when we consider that in this particular class, many of them, from the want of any very absorbing occupation, would have more than the ordinary temptation to indulge in the excesses of such a mode of life—and further, that most of them would be possessed of ample pecuniary means for such indulgence—we need not, I think, look any further for the explanation of the mystery. But whether this conclusion be correct, or not, it is evident that the anomaly cannot be owing to any *constitutional* peculiarity in the *class*, for it is entirely absent from the

corresponding observations on female life. Nor can it arise from any peculiarity in the *sex*, for the males of other observations are free from it.

It may not, perhaps, be out of place to refer here, in passing, to the extraordinary difference which we find between the mortality of male and female life in the observations upon the Government annuitants. The authors of the very valuable observations upon the peerage families suggest that this may perhaps be owing to the probable preponderance of unmarried females in Mr. Finlaison's observations; but I think it important to point out that there are two circumstances which militate against this supposition. The first is, that although doubtless a preponderance of unmarried females would probably be found in the case of annuities purchased at the later periods of life, there is no reason to suppose that the same would be found in the lives admitted at the earlier ages, who, as previously observed, consist for the most part of the nominees in various tontines, many of whom must have been selected in childhood; yet the anomaly is the more remarkable (because unusual) at the earlier adult ages than at the more advanced periods of life. But a yet stronger reason for rejecting the hypothesis is, that by comparing the mortality of each sex with the corresponding mortality of the peerage families, we find that the great difference between the two sexes in the Government annuitants arises not from an excessive vitality in the females, but from an excessive mortality in the males—a phenomenon for which I cannot suggest any very satisfactory explanation, but the cause of which it would, I think, be very desirable to investigate.

It appears to me, then, from the considerations which I have endeavoured to explain, that, as regards the particular observations we have examined, the best data for determining the nature of the normal law of mortality—or of the variation in the rate of mortality, from age to age, which arises solely from the gradual decay of the vital powers—will be found in the males of Assurance and Friendly Societies and the clergy, and in the females of the "Government Annuitants" and "Peerage Families."

The next point which we have to consider is the mode of adjusting the mortality table in accordance with the very simple nature of the progression found in each of the observations last referred to; in other words, of correcting the aberrations (for such, according to the views herein advanced, they must be considered) which prevail to a considerable extent in many very useful and otherwise trustworthy observations, and, indeed, to some extent

in all observations whatever. If our object be confined to this point alone, there is, I conceive, but little remaining to be done. The ablest cultivators of the science of life contingencies have devoted their powers to the elucidation of the principles upon which this should be effected; and in the works of the many eminent writers on life contingencies the student will find a vast and most instructive fund of learning and ingenuity. Most of these writers either tacitly or expressly assume the necessity of preserving an uninterrupted progression in the series representing the annual rate of mortality, the soundness of which assumption I have endeavoured to establish in the foregoing pages.

The methods of adjustment proposed by Messrs. Jellicoe and Gray (whose valuable papers on this subject have appeared in the *Assurance Magazine*) are founded upon the formula devised by Mr. Gompertz; and there can be no question that those gentlemen have satisfactorily shown that the formula in question is admirably adapted for the purpose. We occasionally hear objections expressed to the use of a mathematical formula, as an instrument of adjustment; but admitting the necessity of some process of the kind, it is difficult to see why it should not be effected by such means, provided, of course, that the result is satisfactory. Indeed, to most persons the highly philosophical principle from which Mr. Gompertz's formula is derived is, I think, sufficient to give it a decided preference over every other method.

But so far from considering the use of a mathematical formula objectionable, I look upon it as the best means of correcting the aberrations which are found in a greater or less degree in all observations, whether arising from paucity of numbers or from the existence of abnormal conditions in the data observed; a process which, as I have endeavoured to show, is essential to the proper construction of tables for computing the values of isolated contingencies.

Mr. Gompertz's theory of the law of mortality is, that the vital power, or the "power to oppose destruction," loses equal proportions in equal times; and consequently that the intensity of mortality, which is inversely proportional to this power, is represented by a series in geometrical progression. Now, if we admit the necessity of securing a gradual and progressive increase in the rate of mortality in passing from age to age, it is evident that this theory affords us a very convenient means of adjusting mortality tables; for by taking intervals of sufficient length, an increasing geometrical progression may always be secured. It has, however,

invariably been found that the ratio of progression, instead of remaining constant throughout the whole period of life, as the theory supposes, is, on the contrary, subject to a slow but continued increase with age; in consequence of which it has been found necessary to change the constants at least once, but generally twice, in the construction of a complete table of mortality. But the necessity for this change of constants may be obviated, and the required retardation in the rate of increase, at the earlier ages, may be obtained in as effectual and, I venture to think, in a more scientific manner, by supposing the intensity of mortality to be represented by a series not purely geometrical, but consisting of the sum of two terms, the one a constant quantity and the other geometrical. That is, instead of representing the intensity of mortality by an expression of the form bq^x we represent it by one of the form $c+bq^x$. By this modification of Mr. Gompertz's formula the equation for the numbers living at successive ages becomes $L_x=d.g^{q^x}.s^x$; from which it is perceived that a new constant (s) enters in the formula in a way precisely analogous to that in which the rate of interest is combined with it.

I will now briefly recapitulate the conclusions which follow from the views which I have endeavoured to enforce in the preceding pages.

First, then, it appears that there are two distinct principles to be observed in the adjustment of mortality tables—the one, which merely softens down the asperities of the original observations, without interfering with the nature of the progression at different periods of life; and the other, which imparts an uniformity of progression, consistent with the normal law of mortality, by eliminating the inherent imperfections of the data, whether arising from some of the lives being subject to exceptional causes of mortality at particular periods of life (as in the males of the Government annuitants and peerage families), or from defective means of observation, which in all probability is the case with the Carlisle table, and others similarly formed. The first of these methods is applicable when our object is merely to ascertain the nature of the law of mortality to which the lives *taken in the aggregate, or without selection,* have been and will also, probably, be in future subject; and the second, when we wish to form a standard for the determination of the probabilities of life in the case of *individuals* who are supposed to be at the time in a good state of health, and leading a life conducive to that state.

Secondly, I have endeavoured to show that certain observations

are, from their nature, much better adapted than others to afford a true representation of the normal law of mortality (by which term I mean the nature of the progression in the series denoting the intensity of mortality which is due exclusively to the decay of the vital power), and that it is precisely in those tables that a simple and uniform law of progression is most observable. These tables are—for *males*, the observations upon assured lives, the members of Friendly Societies, and the clergy; and for *females*, the Government annuitants and peerage families.

The third conclusion to which I have arrived is, that from the age of 15, or thereabouts, the normal law of mortality, of which we are in search, is characterized by an increasing progression throughout; the rate of increase, however, being at first very slow, and gradually gaining in rapidity with increased age. It is this characteristic which renders the formula before described (consisting of a constant combined with an increasing geometrical series) singularly well adapted to represent the law in question from adolescence to extreme old age—a satisfactory proof of which assertion I hope to give on a future occasion, when I propose also to examine the results of an extension of the formula to all periods of life.

I wish it to be distinctly understood, nevertheless, that I do not by any means contend that a table, such, for instance, as the males of the Government annuitants or the peerage families, even when corrected by means of the formula which I propose, will form a *perfect* instrument for the purpose of estimating the contingencies of individual selected life. On the contrary, I have stated that it is only by observations on those lives classified according to the age of selection that such a desideratum can be obtained. I only argue that, so corrected, the table will be a better standard than when adjusted upon a narrower or more restricted basis.

Indeed, I incline to the opinion that the selection of the table of mortality best adapted for estimating the value of a particular life must ever be, to a considerable extent, arbitrary, or a matter of opinion. In calculations involving the interest of money we have a few broadly distinguished rates upon which we base our calculations, the particular rate selected being a subject of previous agreement between the parties to the contract. We never think it necessary to make the subdivisions of the rate of interest less than one quarter per cent., and scarcely ever indeed go even to that extent—and this because we know that it is impossible to determine the real future value of money to such a degree of accuracy.

Now, it appears to me that we are situated very similarly with regard to the mortality table; and that for practical purposes what we require is, some three or four broadly marked tables to serve as the standards of value, founded upon the best observations, but corrected as nearly as possible according to what appears to be the natural or normal law of mortality. Let us consider for a moment the probable effect which the different positions of individuals, even in the same class of society (not to mention hundreds of other equally important considerations), must have upon the chances of life. Yet a calculator will proceed to make his valuation without inquiring whether the life proposed is a person who is free from any harassing cares, and has nothing to do but to study his health and comfort, or whether he is burthened with the responsibilities attaching to an onerous and important office; nor could he do much with the information if he obtained it. To stipulate then, as an essential condition, for a calculation in strict accordance with the data supplied by any set of observations, may bear an *ad captandum* appearance of accuracy; but, from what I have stated, such an appearance must be entirely illusory, and such a stipulation is, I think, scarcely consistent with a philosophical or scientific view of the subject.

I must, however, guard myself against the risk of being supposed to underrate the value of tables showing the mortality prevailing in specific classes of the community—such, for instance, as the "Peerage Families" and the "Clergy." Such tables I consider of the greatest possible importance, as indicating the result upon the value of life of the peculiar conditions which affect different classes of society; and we are under the deepest obligation to the ingenious authors of these valuable tables. Neither can there be the slightest objection to the use of such tables in computing the pecuniary values of contingencies upon lives belonging to these respective classes; and, indeed, when our information respecting the life is restricted to the condition which forms the basis of the table, such table will indicate precisely the true value of the given contingency. But as it will seldom happen that our information *is* so restricted—as we shall generally know something about the individual which will induce us to think that the true value of the life must differ from the value indicated by the table quite as much as the best tables are found to differ from one another—it is, I think, in the former rather than in the latter capacity, as guides rather than as instruments of calculation, that the chief value of these tables will be found to consist.

It is, perhaps, from some such view of the matter as that which I am endeavouring to explain that we must account for the hold which the Carlisle Table still retains as a measure of the value of life contingencies in this country. With all its faults of adjustment it has been found to agree *in the main* with observations founded upon much better principles and upon a more extended basis; and it is felt that having been so generally adopted as a standard it would be but an idle *affectation* of refinement to discard it and the many useful tables which have been formed upon it, for tables of the latter description. It is for this reason that, in the construction of the table which I now submit for the consideration of the members, I have adopted the Carlisle table as a basis, and I now proceed to show that the alterations which I have introduced by the process of readjustment are merely corrections of the defects of that celebrated table.

I may here observe that Mr. Milne's adjustment of the Carlisle table is, in my opinion, a striking instance of the misapplication of the first of the two methods described in the foregoing pages. The table was intended, not simply as a representation of the law of mortality which had prevailed amongst the particular community observed, but for the far more important object of furnishing a better standard for the valuation of life contingencies than any then in use. Instead, therefore, of adopting the method of adjustment first described, whereby the final table exhibits all the peculiar and abnormal features of those particular observations—and which, as I shall show, are in all probability due to errors in the mode of collecting the data—Mr. Milne should have kept to the broad and simple features common to *all* observations, as described by Dr. Price, and as exemplified in all the tables then in use. His table would then have assumed somewhat of the form of the readjusted table which I have constructed by the use of the formula herein described.

I cannot help thinking that Mr. Milne must himself have been partly sensible of the distinction which I have endeavoured to draw between these two modes of adjustment; for while he has been careful to preserve in his mortality table all the irregularities which his data exhibited, yet I cannot find that he computed his tables of premiums for "term" assurances (in which alone these irregularities would be visible) upon the same principle. In the tables of premiums published at the end of the second volume of Mr. David Jones' work, I find that the rates of the Sun Office for assurances for a single year are in increasing progression through-

out. Now Mr. Milne could hardly have been so inconsistent as to hold that it is necessary to adhere strictly to the data in the case of "whole life" premiums—where the results of adjustment are comparatively insignificant—and to abandon them in the case of "term" premiums, in which alone the process of adjustment makes any material difference. It is more reasonable to suppose that he felt the inapplicability of his method of adjustment to a table intended for the calculation of assurance premiums.

But a stronger argument in favour of a readjustment of the Carlisle table will, I think, be found in the following somewhat curious circumstance connected with the construction of that celebrated table, which I have never yet seen noticed. From Mr. Milne's observations on the method of constructing mortality tables, in his first volume, and his explanation respecting the application of the method to the Carlisle table, in the second, we may, I think, conclude that he believed Dr. Heysham's observations upon the inhabitants of Carlisle in the years 1780 and 1787 were founded in each case upon enumerations of the numbers living in the several stated periods, and not of the total number of the population only. With regard to the enumeration of 1780, which was made by Dr. Heysham himself, that gentleman says (Milne, vol. ii., p. 746), "When I made the survey of Carlisle, in the beginning of the year 1780, there were between [the ages of] 10 and 15, 715; and between 15 and 20 years of age, 675," &c. From this we may infer that the survey of 1780 comprised a complete enumeration of the ages as well as the number of the inhabitants; but is it equally clear that this was so with the survey of 1787?

The latter enumeration was made in pursuance of an order from the Court of Quarter Sessions to the different constables in the county of Cumberland, "to make an actual survey of all the inhabitants of the county." This is pretty nearly all the information given on the subject in Mr. Milne's abridgment of Dr. Heysham's pamphlet; and nowhere is it therein stated whether the enumeration took account of the ages of the inhabitants. The original pamphlet of Dr. Heysham I have not the means of consulting.

The first three columns of the following table contain the data upon which Mr. Milne determined the numbers exposed to the risk of mortality during the eight years comprised in Dr. Heysham's observations.

III.—*Population of Carlisle.*

Between the Ages of	Number Living in		
	Jan., 1780.	Dec., 1787.	
(1)	(2)	(3)	(4)
0 and 5	1,029	1,164	1,163
5 „ 10	908	1,026	1,026
10 „ 15	715	808	808
15 „ 20	675	763	763
20 „ 30	1,328	1,501	1,501
30 „ 40	877	991	991
40 „ 50	858	970	970
50 „ 60	588	665	665
60 „ 70	438	494	495
70 „ 80	191	216	216
80 „ 90	58	66	66
90 „ 100	10	11	11
100 „ 105	2	2	2
Total	7,677	8,677	8,677

The first column contains the several intervals of age; the second, the numbers living in each interval in January, 1780; and the third, the numbers living in December, 1787; the fourth and last column, which I have added myself, is derived from the second, by increasing each item in the exact proportion in which the total population had increased during the interval, that is, I have multiplied each item in column 2 by $\frac{8677}{7677}$.

By a comparison of this last column with the one immediately preceding it, I hold it to be a fair assumption (until positive evidence of the contrary be produced) that the latter was obtained, not by an enumeration of the inhabitants according to age, but by the same method as that by which I have constructed the fourth column. The two instances in which the results differ (in each case by unity only) are, perhaps, just sufficient to show that the proportions were not determined by an actuary.

Dr. Heysham estimates that of the increase of 1,000 inhabitants (in a population of 7,677), which took place during the period observed, more than one-half (viz., 511) arose from an excess of new settlers over emigrants. The actual disturbance in the population is very inadequately represented by the number in question, which is the *difference* only between the numbers of immigrants and emigrants; the actual number of the former must have been

much greater. Now, these opposing forces, it is assumed in the construction of Mr. Milne's table, were so nicely regulated as to distribute the excess in the exact proportion to the ages of the existing population. May we not, therefore, fairly suspect that the irregularities of the Carlisle table may be in some measure attributable to the perverseness of these two bodies—the new settlers and the emigrants—in not regulating their movements in accordance with so convenient an hypothesis?

Proceeding now with the explanation of the modifications which I have introduced by my process of readjustment, I have to beg attention to the following table, which exhibits in one view the original and readjusted Carlisle table, together with the results of the principal observations previously examined; the latter, for convenience of comparison being combined as follows:—Males of the Government annuitants with those of the peerage families, males of the Assurance Office with those of the Friendly Societies, and females of the Government annuitants with those of the peerage families.

IV.—*Annual Mortality per* 1,000.

Age.	Government Annuitants and Peerage. (Males.)	Carlisle.	Ditto, Readjusted.	Friendly Societies and Assured Lives. (Males.)	Government Annuitants and Peerage. (Females.)	Age.
18	9·2	7·0	7·6	7·6	8·3	18
23	13·4	7·0	8·0	7·9	8·5	23
28	11·8	8·7	8·5	8·4	8·9	28
33	9·5	10·1	9·3	8·7	10·3	33
38	12·3	11·2	10·4	10·4	10·8	38
43	12·5	14·6	12·3	12·1	11·6	43
48	14·4	13·9	15·0	15·4	13·7	48
53	19·8	16·1	19·3	19·9	15·6	53
58	24·5	24·2	25·7	28·8	20·8	58
63	37·0	38·3	35·6	35·2	32·0	63
68	56·4	46·5	50·4	50·9	40·6	68
73	80·1	78·1	72·7	72·8	61·8	73
78	109·3	108·8	105·8	111·4	88·4	78

Upon examining the above table we remark that the deviations from the normal law of progression observable in the Carlisle data are mostly of an entirely opposite character to those of the Government annuitants and peerage families (males). Thus, instead of an undue increase from 18 to 23, followed by a decrease until 33, we find, on the contrary, that the first abnormal increase commences at 28 and continues until 33. Again, in the Government annuitants and peerage families (males) a rapid increase takes place from 33 to 38, followed by a suspended mortality in the

next five years. In the Carlisle table, on the contrary, the increase is normal from 33 to 38, but an undue increase occurs from 38 to 43, followed by a decrease in the next five years. These inconsistencies in Mr. Milne's table I have endeavoured to show are, in all probability, due to defective means of observation; and if so, it follows that the very great pains which that able author appears to have taken in the adjustment of his table were not only unnecessary, but have rendered it far less correct (even when considered merely as a representation of *physical facts*) than if he had adopted the simpler and less laborious processes used by his predecessors.

It will also be seen that the readjusted Carlisle table follows very closely the mean of the "Assured Lives" and "Friendly Societies." The only ages at which there is any material difference are 33, 58, and 78; and it will be seen that at these ages the readjusted Carlisle table follows more nearly than the "mean" table of "Assured Lives" and "Friendly Societies" the general run of the *other* observations; from which it is not unreasonable to infer that the "mean" table in question is to some extent abnormal at these particular ages. Thus, at age 33, where the "mean" is *below* the "readjusted" table, it is also below every other observation; at 58, where it is considerably *above*, it is still more in excess of every other; and again, at 78, where it is also *above*, it will again be found to be higher than all the others.

It may not be out of place to refer here to the great superiority in point of vitality of the males of "Friendly Societies" over those of the "Assured Lives." The explanatory remarks prefixed to the observations on "Assured Lives" are altogether silent on the subject of the mode in which lives taken at special rates of premium —whether on account of deteriorated health or foreign residence— were treated; from which, I presume, we must infer that no distinction was made between these and other cases. If so, this would doubtless account, to some extent, for the superiority in question. Further, there can, I think, be little doubt that owing to the withdrawal of many of the best lives by the surrender and discontinuance of policies, the mortality among assured lives is higher than it would have been if the whole of the members had remained on the books, while in the Friendly Societies the discontinuance of membership is supposed to have a contrary effect. Nor is this seeming contradiction (if it exists) impossible to explain. The member of an Assurance Society, if his health fails him, will exhaust every means to keep up his policy; and should he find it impossible to accomplish this, he will have little difficulty in finding

a purchaser, among the many speculators in such property, upon better terms than the rules of the Office will afford him. The member of a Friendly Society, on the other hand, if he becomes unable to continue his payments, as *his* policy is not a marketable commodity, has no alternative but to forfeit his claim upon the Society. Again, it is no unusual thing to find, when a member of an Assurance Society becomes invalided, destitute, or of intemperate habits, the friends of the wife will keep up the payments, if only to prevent the family becoming a burthen upon them. Such is not, I apprehend, generally speaking, the case with the member of a Friendly Society. His friends, if they are as willing, are less able to help him in his necessity. Besides, a considerable portion of his subscription, I believe, is expended in the enjoyments of convivial meetings; and thus his membership partakes of the nature of a luxury. His friends therefore, even if they were as able, would on this account be less disposed to tax themselves in order to maintain it.

As an illustration of the closeness with which the formula proposed, while, as I have endeavoured to show, it improves the series exhibiting the rate of mortality from age to age, at the same time preserves the *general effect* of the observations dealt with, I submit the following table of the mean duration of life by the Carlisle table and also by the readjusted table. It will thus be seen that the greatest difference between the two (under the age of 88, beyond which I consider any comparison useless) is ·35, occurring at ages 47 and 48. Now, if we refer to Mr. Gray's very able paper, in vol. vii. of the *Assurance Magazine*, wherein that gentleman gives a table constructed by the application of Mr. Gompertz's formula to the same table, we find the greatest divergence to be ·28, which occurs at the ages 66 and 67. But the interval taken by Mr. Gray is 10 years only, which he is enabled to do by changing the constants *twice* during the process. The interval I am compelled to adopt, owing to the use of a formula with four *unchanged* constants, is 25 years, or two and a half times as great as Mr. Gray's; while the greatest divergence in my table exceeds that of Mr. Gray's by only one-fourth. And looking at it merely as a question of the comparative closeness of the agreement, I think I may say that the "very awkward break" which Mr. Gray points out as occurring in his table at age 50, is got rid of in my table with no very great sacrifice of approximate accuracy. In Mr. Jellicoe's excellent adjustment of the Eagle Experience, the greatest difference between the unadjusted and the adjusted results

is ·76, which occurs at age 22. In this case, however, it must be observed that there was no preliminary adjustment of the rough data previous to the application of the formula used by Mr. Jellicoe, as in the case of the Carlisle table. I need scarcely add, that my object in making these comparisons is to show that, considering my formula simply as an instrument for *adjusting* tables of mortality, the alterations of the original data are not significantly greater than those introduced by the methods adopted by the most eminent authorities.

V.—*Mean Duration of Life.*

Age.	Carlisle Table.	Same Readjusted.	Age.	Carlisle Table.	Same Readjusted.	Age.	Carlisle Table.	Same Readjusted.
15	45·00	44·88	46	23·82	23·50	77	6·40	6·41
16	44·27	44·22	47	23·17	22·82	78	6·12	6·05
17	43·57	43·55	48	22·50	22·15	79	5·80	5·71
18	42·87	42·88	49	21·81	21·48	80	5·51	5·38
19	42·17	42·21	50	21·11	20·81	81	5·21	5·07
20	41·46	41·53	51	20·39	20·16	82	4·93	4·77
21	40·75	40·85	52	19·68	19·50	83	4·65	4·48
22	40·04	40·17	53	18·97	18·86	84	4·39	4·21
23	39·31	39·48	54	18·28	18·22	85	4·12	3·95
24	38·59	38·80	55	17·58	17·59	86	3·90	3·70
25	37·86	38·11	56	16·89	16·96	87	3·71	3·47
26	37·14	37·42	57	16·21	16·35	88	3·59	3·24
27	36·41	36·72	58	15·55	15·74	89	3·47	3·03
28	35·69	36·03	59	14·92	15·14	90	3·28	2·83
29	35·00	35·33	60	14·34	14·56	91	3·26	2·65
30	34·34	34·64	61	13·82	13·98	92	3·37	2·47
31	33·68	33·94	62	13·31	13·41	93	3·48	2·30
32	33·03	33·24	63	12·81	12·86	94	3·53	2·15
33	32·36	32·54	64	12·30	12·31	95	3·53	2·00
34	31·68	31·84	65	11·79	11·78	96	3·46	1·87
35	31·00	31·14	66	11·27	11·26	97	3·28	1·74
36	30·32	30·44	67	10·75	10·75	98	3·07	1·62
37	29·64	29·73	68	10·23	10·25	99	2·77	1·51
38	28·96	29·03	69	9·70	9·77	100	2·28	1·41
39	28·28	28·34	70	9·18	9·30	101	1·79	1·31
40	27·61	27·64	71	8·65	8·85	102	1·30	1·22
41	26·97	26·94	72	8·16	8·41	103	·83	1·14
42	26·34	26·25	73	7·72	7·98	104	·50	1·07
43	25·71	25·56	74	7·33	7·57	105	..	1·00
44	25·09	24·87	75	7·01	7·17	106	..	·94
45	24·46	24·18	76	6·69	6·78	107	..	·88

Reverting now to Mr. Higham's paper on the effect of selection among assured lives, while I admire the skill displayed in the treatment of his data I must express my decided dissent from some of the conclusions which he draws from them; such, for instance, as the justification which he imagines he finds in his results for the use of the Northampton 3 per cent. table in the calculation of premiums for assurance. Mr. Higham, it appears to me, has not

considered in its proper light an important element in the observations on assured lives, the existence of which renders such observations unsuitable for the determination of rates of premium, whether taken in the aggregate or in separate classes according to the age at admission. To render them suitable for this purpose it would be necessary, after classifying the lives in the way referred to, to trace each life during the remainder of the period observed. But it is a well known fact that many lives are prematurely withdrawn from observation by the discontinuance or surrender of the policy, and such withdrawals necessarily consist of *select* lives—that is, of course, select at the time of withdrawal—for, as previously observed, the deteriorated and doubtful lives do not withdraw.* This constant draining of the better class of lives must necessarily have the effect of materially increasing the rate of mortality among the lives which remain upon the books; a result which, indeed, is sufficiently evident from the following comparison between the mean duration of life according to Mr. Higham's "Class Mortality," where the lives are *selected* at the given age, and also according to Mr. Milne's Carlisle table.

VI.—*Mean Duration of Life.*

Age.	Assured Lives.	Carlisle.
25	36·50	37·86
30	33·48	34·34
35	30·38	31·00
40	27·19	27·61
45	23·77	24·46
50	20·49	21·11
55	17·35	17·58
60	14·54	14·34
65	11·83	11·79
70	9·47	9·18
75	7·62	7·01

Here we see that at the age 60 and *upwards* the "class" mortality is in every instance more favourable than the Carlisle table, while going *backwards* towards the younger ages we find the contrary result—the vitality of the "class" observations being below the Carlisle, and the difference increases as we approach the younger ages. Now, what ought we to infer from this fact? Evidently, *not* that the lives admitted at the age 25, for instance,

* The observations on the "Government Annuitants," if the elementary facts were accessible to the public, would furnish the means of testing the effect of selection without the drawback here adverted to.

were not of that select description at the time of admission which we suppose them to be, but that their aggregate vitality has been reduced by the constant drain of the best lives, which has been going on since the time of admission by the discontinuance and surrender of policies.

This, indeed, is the conclusion arrived at by Mr. Higham, and so far, therefore, I perfectly agree with him. But he goes on to argue that we ought therefore to calculate our rates of premium accordingly. That is, in order to provide for the loss occasioned by the probable withdrawal of *some* of the lives, an additional tax should be levied, from the commencement, upon the *whole*. This, of course, is *one* way of meeting the difficulty, but it is not, in my opinion, the proper way.

The only safe and equitable plan in matters of this kind is, to follow the simple rule which Mr. Gompertz so strongly insists upon in his paper, read before the Royal Society in June, 1820, viz., that we must first make our estimate according to the real facts of the case, and upon the most *accurate* elements procurable, both as regards the rate of interest and the rate of mortality, and then make a margin for security and contingencies, in whichever direction the same may be necessary. Thus, in calculating the rate of premium, we must treat the life (as it really is) as a select life, *adding* the margin for contingencies; and in valuing the policy for surrender, we must also proceed upon the assumption of the life being *then* a select one, *deducting* in this case the margin for contingencies. By this arrangement the retiring policyholder pays (as he ought to do) for the disturbance in the average quality of the remaining lives which he occasions. If Mr. Gompertz's sound and rational maxim had been earlier understood and acted upon, we should never have heard of the preposterous terms which some Offices even yet allow as an inducement for their best lives to leave them, thereby inflicting an injury, not upon themselves only, but upon the family of the surrendering policyholder, who might otherwise be induced to make an effort to keep up his payments.

In reference to this subject, however, it is important to bear in mind, that whichever mode we may adopt to protect the Office against the *loss* occasioned by the deterioration of its lives by the surrender of policies, yet the fact of the deterioration remains, and must therefore be taken into account in making an estimate of the outstanding liabilities of an Office. Great judgment and discrimination are therefore evidently required in determining the table of mortality to be used in a distribution of profits, the more so as a very great

difference in the amount of the estimated surplus will result from the use of different tables for this purpose.

In conclusion, I have merely to add, that out of different modifications of Mr. Gompertz's formula which I have tried for the purpose of adjustment, I have been determined in my selection of the one herein proposed chiefly by the fact of its affording a very important aid in the construction of tables for annuities and assurances involving two or more lives. This is effected by means of a property analogous to that of "uniform seniority," which Professor De Morgan has shown to belong to the formula of Mr. Gompertz, and which latter property is described as follows:—If x and $x+h$ represent the ages of two given lives, then the annuity on the joint existence of those lives will be precisely equal to an annuity on a single life whose age is $x+\dfrac{\log(1+q^h)}{\log q}$. In the modification of Mr. Gompertz's formula which I have adopted the same law holds good; but instead of being equal to an annuity on a single life, the required joint life annuity is equal to an annuity on two equal joint lives whose common age is $x+\dfrac{\log\dfrac{1+q^h}{2}}{\log q}$. And generally the annuity on any number (n) of joint lives, aged respectively $x, x+h, x+k, x+l \ldots$, will be equal to an annuity on the same number of equal joint lives whose common age is $x+\dfrac{\log\dfrac{1+q^h+q^k+q^l+\ldots}{n}}{\log q}$.
It will not, however, be necessary (in practice) to calculate the value of this expression, as, by means of the equation $q^x+q^{x+h}+q^{x+k}+q^{x+l}+\ldots=nq^r$ (where r is the equivalent common age), and tables of annuities for every value of nq^r, we have a simpler method of obtaining the value of the required annuity. The full elucidation of this principle, and the advantages afforded by it, will form the subject of another paper, which I hope to have the honour, on some future occasion, of reading before the members of this Institute.

Having thus endeavoured to establish the principles which it appears to me should always be kept in view in constructing tables of mortality, I propose, in the next division of my subject, to exemplify those principles by practical applications of a method of graduation to some of the preceding tables. The third and concluding portion will comprise an explanation of the method of constructing tables for solving the various problems in life contingencies upon the basis of the property mentioned in the preceding paragraph.

*On the Calculation of Single Life Contingencies.** (*Part I.*) *By* PROFESSOR DE MORGAN.

IT is the object of the present article to put together a number of formulæ which it may be useful to the actuary to find in one place. At the same time it may show all persons who possess an elementary knowledge of algebra, that they may, with no great amount of tables, and processes of very easy application, learn to compute the value of any benefit in which the duration of one life only is concerned. The same principles, with more extensive tables, apply to cases in which two or more lives are involved.

A sketch of the history of the subject will be found in the Library of Useful Knowledge, Treatise on *Probability;* and more fully in Mr. Milne's articles on *Mortality* and *Annuities* in the new edition of the Encyclopædia Britannica. The articles *Annuity, Interest, Mortality,* and *Reversions* (when the latter appears) in the Penny Cyclopædia, may also be consulted.

About thirty years ago, a Mr. George Barrett presented to the Royal Society a method by which the calculation of life contingencies was very materially facilitated. This method the society did not think worthy of publication; and it was accordingly given to the world by Mr. Francis Baily, in the appendix to his well-known work on Annuities, with some severe remarks on the omission just alluded to. It was certainly an unfortunate want either of examination or of judgment which caused the Philosophical Transactions, the depositary of the writings of many eminent inquirers on this subject, to miss a contribution which would have done honour to any one of them. This method of Mr. Barrett was rendered still more commodious, and we believe, extended, by Mr. Griffith Davies, in his *Tables of Life Contingencies* (1825), a work now unfortunately out of print: it is Mr. Barrett's method, as improved by Mr. Davies, which we propose to present to the reader, with some extension of notation and generalization of processes.

Let it be the law of mortality that of a_0 persons born alive,

* It has been suggested that a reprint in the *Journal* of this paper, which, it will be remembered, first appeared in the pages of the *Companion to the British Almanac* for 1840, would be very desirable on many grounds; and entirely concurring in the suggestion, we now place it before our readers, having obtained the needful authority to do so. We have the satisfaction to add that Professor De Morgan has corrected the proofs of this *reprint*, which name strictly applies, nothing being altered except obvious misprints.—ED. *A. M.*

a_1 are alive at the end of a year, a_2 at the end of two years, a_x at the end of x years. Let v be the present value of £1, to be received at the end of a year, which depends entirely on the interest of money; if r be the interest of £1 for one year, we have

$$v=\frac{1}{1+r}:$$

thus, at 3 per cent., $r=\cdot03$ and $v=\cdot970874$. The present values of £1 to be received at the end of 2, 3, ... x years are v^2, v^3, ... v^x. The best way to find these powers will be to take the logarithms* from a larger table, such as that in the Penny Cyclopædia, article Interest. Thus the logarithm of $1+r$ being

$\cdot0128372247$, that of v is $9\cdot9871627753-10$.

Multiples of this logarithm being formed up to 104 times, we have all the logarithms which will be wanted in the use of the Carlisle table.

Persons not used to computation should remember that the easiest way of forming a set of multiples is to write the quantity to be added each time at the bottom of a card, and to make each addition by holding the card so that the writing on it may stand over the last result. In this way it will not take many minutes to form a hundred multiples of the preceding, and a verification of the last multiple should be made by actual multiplication.

Obtain as many logarithms as are wanted of the powers of v in the preceding manner, allowing as much space between the lines as will† contain four lines of figures. Take the table of mortality which is to be used, say the Carlisle table, and under the logarithm of v write that of a_1, the number surviving a year; under that of v^2, write the logarithm of a_2; and so on up to the end of life. The Carlisle table will be found in Mr. Milne's work on Annuities; it is also in the Penny Cyclopædia, article‡ *Mortality*. Under the last logarithms write in succession the logarithms of a_0-a_1, a_1-a_2, &c., the numbers who die in the first, second, &c. years: as follows, in which 3 per cent. is supposed. Five decimal places are taken, merely as an example; but it will be almost as easy to use seven, as there are no interpolations.

* We have re-examined this table, and find no error, by Hutton's Tables, p. 386.

† It will be desirable to have the spaces equal.

‡ We have re-examined this reprint, and find it correct.

	10000			
1		$\log v$	$=9{\cdot}98716-10$	
2	8461	$\log a_1$	$=3{\cdot}92742$	
3	1539	$\log (a_0-a_1)$	$=3{\cdot}18724$	
4			$3{\cdot}91458$	$a_1v=8214{\cdot}5$
5			$3{\cdot}17440$	$(a_0-a_1)v=1494{\cdot}2$
1		$\log v^2$	$=9{\cdot}97433-10$	
2	7779	$\log a_2$	$=3{\cdot}89092$	
3	682	$\log (a_1-a_2)$	$=2{\cdot}83378$	
4			$3{\cdot}86525$	$a_2v^2=7332{\cdot}5$
5			$2{\cdot}80811$	$(a_1-a_2)v^2=6428{\cdot}5$
	&c.	&c.	&c.	&c. &c.

Those who cannot easily add one line to another which is separated by a third should now cut pieces out of a card in such manner that being laid upon one of the compartments of the table, the parts cut open will show the first and third line, and the rest hide the second and fourth. The fourth line is then formed by adding the first and second, and the fifth line by adding the first and third, covering the second and fourth. We thus obtain two successions of results:—

$$\begin{array}{ccccc} a_0 & a_1v & a_2v^2 & a_3v^3, & \&c. \\ (a_0-a_1)v & (a_1-a_2)v^2 & (a_2-a_3)v^3 & (a_3-a_4)v^4, & \&c. \end{array}$$

Let these be denominated D_0, D_1, D_2, &c., and C_0, C_1, C_2, &c., so that

$$D_0=a_0, \qquad D_1=a_1v, \ldots\ldots\ldots D_x=a_xv^x$$
$$C_0=(a_0-a_1)v, \quad C_1=(a_1-a_2)v^2, \ldots\ldots C_x=(a_x-a_{x+1})v^{x+1}.$$

The following table is then to be completed in the manner which will be described.

Age.	D_x	N_x	S_x	C_x	M_x	R_x	Age.
0	D_0	N_0	S_0	C_0	M_0	R_0	0
1	D_1	N_1	S_1	C_1	M_1	R_1	1
2	D_2	N_2	S_2	C_2	M_2	R_2	2
3	D_3	N_3	S_3	C_3	M_3	R_3	3
&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.

The columns D and C have been described; the rest are formed from them as follows. The last of the column N is nothing, and N_x is always the sum of those in column D beginning with D_{x+1}, and continuing to the end: thus any one D and its N added together give the preceding N. Or

$$N_x=D_{x+1}+D_{x+2}+D_{x+3}+\ldots\ldots(\text{to the end})$$
$$N_x=D_{x+1}+N_{x+1}.$$

The last S is nothing, and the column S is formed from N as N was formed from D in every point except this, that each S begins with its own N instead of the one after: thus

$$S_x = N_x + N_{x+1} + N_{x+2} + \ldots\ldots \text{(to the end)}$$
$$S_x = N_x + S_{x+1}.$$

The last M is the last C, and M is formed from C precisely as S from N: thus—

$$M_x = C_x + C_{x+1} + C_{x+2} + \ldots\ldots \text{(to the end)}$$
$$M_x = C_x + M_{x+1}.$$

Lastly, R is formed from M as M from C: thus

$$R_x = M_x + M_{x+1} + M_{x+2} + \ldots\ldots \text{(to the end)}$$
$$R_x = M_x + R_{x+1}.$$

We here give, as a specimen, the first five, the last five, and an intermediate five years, from the Carlisle table at 3 per cent., keeping only four significant figures :—

Age.	D.	N.	S.	C.	M.	R.	Age.	No. Living.	Dying.
0	10000	173200	3702000	1494·0	4664	70040	0	10000	1539
1	8215	165000	3529000	642·9	3170	65372	1	8461	682
2	7332	157700	3364000	462·1	2527	62202	2	7779	505
3	6657	151000	3206000	245·2	2065	59674	3	7274	276
4	6218	144800	3055000	173·4	1820	57610	4	6998	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
30	2324	45460	732100	22·80	932·7	25070	30	5642	57
31	2234	43220	686600	22·14	909·9	24140	31	5585	57
32	2147	41080	643400	21·11	887·8	23230	32	5528	56
33	2063	39020	602300	20·13	866·6	22340	33	5472	55
34	1983	37030	563300	19·55	846·5	21470	34	5417	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
100	·4683	·7879	1·4580	·10100	·43170	1·17700	100	9	2
101	·3536	·4343	·6696	·09809	·33070	·74550	101	7	2
102	·2452	·1891	·2353	·09524	·23260	·41480	102	5	2
103	·1429	·0462	·0462	·09246	·13730	·18220	103	3	2
104	·0462	·0000	·0000	·04488	·04488	·04488	104	1	

That the beginner may attach definite ideas to the several columns, we subjoin an explanation of each. The assumption is, that of 10,000 individuals born alive, the numbers surviving to each age, and dying in each year, are as in the last two columns.

D. £2324 invested at 3 per cent. at the birth of 10,000 persons, will, improved at compound interest, yield every survivor £1 at the age of 30: the number of survivors being 5642.

N. £45460 invested at the birth of 10,000 individuals will produce, by the time they attain the age of 30, enough to guarantee

to each person then surviving an annuity of £1 on his life, the first payment being made when he attains the age of 31.

S. £732100 will in the same case produce enough to guarantee an increasing annuity, paying £1 to each at the age of 31, £2 at that of 32, &c.

C. £22·8 invested at 3 per cent. at the birth of 10,000 persons, will, improved at compound interest until the survivors are 31 years of age, yield £1 for each of those who died between 30 and 31.

M. £932·7 similarly invested, would yield to each one who reaches 30, £1 at the end of the year in which he dies.

R. £25070 similarly invested, would yield to each one who lives to be 30, £1 if he die in the 31st year, £2 if in the 32nd, and so on.

The preceding list simply enunciates the method of constructing the tables; the following shows the use to which the mere inspection may be put. Take any two ages, say 30 and 34, and transpose the numbers opposite each age to the other age: then, whatever may be the present age (less than 30)—

D. A person might now give up £1983, due at the age of 30, to receive £2324, if he live to be 34.

N. A person might now give up an annuity of £37030, to be granted at the age of 30, to receive in return another of £45460, to be granted at the age of 34, if he should live so long.

S. A person might now give up a uniformly increasing annuity of £563,300 the first year, twice as much the second, &c., to be entered upon at the age of 30, to receive another annuity of the same kind, beginning with £732,100, to be entered upon at the age of 34, if he should live so long.

C. £19·55 secured to a person in the event of his dying between 30 and 31, is now of the same value as £22·80, secured to the same person in the event of his dying between 34 and 35.

M. £846·5 secured to a person at the end of the year in which he dies, if after attaining 30, is of the same value as £932·7 secured to the same person at the same period, if after attaining 34.

R. An increasing assurance, to be £21470, if a person die in his 31st year, twice as much if in his 32nd, &c., is now of the same value as another, to be £25,070 if he die in his 35th year, twice as much if in his 36th, &c.

These properties are independent of the present age of the party, and show that the most simple indication of the tables is the proportion in which a benefit due at one age ought to be

changed, so as to retain the same value and be due at another age. They might, therefore, with great propriety be called *commutation tables*.

The following formulæ and additional notation will be found useful.

I. $N_x=D_{x+1}+D_{x+2}+D_{x+3}+\ldots$ to the end of life.

$M_x=C_x+C_{x+1}+C_{x+2}+\ldots \quad . \quad . \quad . \quad . \quad .$

$S_x=N_x+N_{x+1}+N_{x+2}+\ldots=D_{x+1}+2D_{x+2}+3D_{x+3}+\ldots$

$R_x=M_x+M_{x+1}+M_{x+2}+\ldots=C_x+2C_{x+1}+3C_{x+2}+\ldots$

II. $N_x-N_{x+1}=D_{x+1}$ $\qquad$ $S_x-S_{x+1}=N_x$

$M_x-M_{x+1}=C_x$ $\qquad$ $R_x-R_{x+1}=M_x$

III. Let $N_{x,y}=N_x-N_{x+y}$ $\quad|\quad$ $S_{x,y}=S_x-S_{x+y}-yN_{x+y-1}$

$M_{x,y}=M_x-M_{x+y}$ $\quad|\quad$ $R_{x,y}=R_x-R_{x+y}-yM_{x+y-1}$

IV. Then $N_{x,y}=D_{x+1}+D_{x+2}+\ldots+D_{x+y}\ \ldots(y \text{ terms})$

$M_{x,y}=C_x+C_{x+1}+\ldots+C_{x+y-1}\ldots(y \text{ terms})$

$S_{x,y}=D_{x+1}+2D_{x+2}+\ldots+(y-1)D_{x+y-1}\ldots(y-1 \text{ terms})$

$R_{x,y}=C_x+2C_{x+1}+\ldots+(y-1)C_{x+y-2}\ldots(y-1 \text{ terms})$

V.
$$D_x+\ldots+D_y=N_{x-1,y-x+1}$$
$$C_x+\ldots+C_y=M_{x,y-x+1}$$
$$D_x+\ldots+(y-x+1)D_y=S_{x-1,y-x+2}$$
$$C_x+\ldots+(y-x+1)C_y=R_{x,y-x+2}$$

VI.
$$N_{x,y}+S_{x+1,y}=S_{x,y+1}=S_x-S_{x+y}-yN_{x+y}$$
$$M_{x,y}+R_{x+1,y}=R_{x,y+1}=R_x-R_{x+y}-yM_{x+y}$$

VII.
$$N_{x,y}-\frac{1}{y}S_{x+1,y}=N_x-\frac{1}{y}(S_{x+1}-S_{x+y+1})$$
$$M_{x,y}-\frac{1}{y}R_{x+1,y}=M_x-\frac{1}{y}(R_{x+1}-R_{x+y+1})$$

VIII.
$$C_x=vD_x-D_{x+1},\quad M_x=vN_{x-1}-N_x$$
$$R_x=vS_{x-1}-S_x$$
$$M_{x,y}=vN_{x-1,y}-N_{x,y},\quad R_{x,y}=vS_{x-1,y}-S_{x,y}$$

The formulæ VIII. will be useful in verifying the tables.

All that will be found of demonstration in the present article is intended for those who are familiar with the subject, being meant to give a method of dealing with the more complicated cases, and particularly a method by which the succeeding formulæ may be verified. This will be followed by a collection of preparations for formulæ which may be easily used even by a person unacquainted with the demonstration.

The present value of £1 to be received by a person now aged x, if he live to attain $x+k$, is $D_{x+k} \div D_x$.

The present value of £1 to be received by the representatives of a person now aged x, if he die between the ages of $x+k$ and $x+k+1$, is $C_{x+k} \div D_x$.

The following problem will include every case we have yet seen proposed of annuities, whether for the whole life, or temporary, or deferred, increasing or decreasing uniformly; and also of insurances: with every manner yet proposed of paying the premium. It matters nothing that it involves payment of premium after the benefits begin to be received, since every application of it will require the part of the premium so paid to be made equal to nothing.

PROBLEM.—A person now aged x is to receive s (pounds sterling) if he attain $x+k$. In the n following years he is, if he live, to receive $a, a+h, \ldots a+(n-1)h$ pounds at the end of the successive years, and ever afterwards during his life, t pounds at the end of each year. But if he die during the n years, he is to have A, A+H .... A+$(n-1)$H, according as he dies in the first, second, &c. year, and T at the end of any subsequent year in which he dies. Besides this, he is, if he die in w years, to have a return of part of the premiums presently described.

For this he is to pay at once σ, and a premium ϖ, which he is to pay l times; the next m premiums are to be ϖ, $\varpi(1+\mu)+\beta$, $\varpi(1+2\mu)+2\beta, \ldots \varpi(1+\overline{m-1}\mu)+\overline{m-1)}\beta$; after which the premium is always τ. But if he should die before he attains $x+w$ years, he is to receive at the end of the year of his death $\rho+\nu\varpi$ if he die in the first year, $\rho+\theta+2\nu\varpi$ if in the second, and, finally, $\rho+(w-1)\theta+w\nu\varpi$ if in the wth year. Required the equation that must exist among these quantities to make the receipts and payments of equal value.

The first set of receipts has the value of the following expression divided by D_x.

$$\begin{aligned}&aD_{x+k+1}+(a+h)D_{x+k+2}+\ldots\ldots+(a+\overline{n-1}h)D_{x+k+n}\\&+AC_{x+k}+(A+H)C_{x+k+1}+\ldots\ldots+(A+\overline{n-1}H)C_{x+k+n-1}\\&+sD_{x+k}+t(D_{x+k+n+1}+\ldots\ldots)\quad+T(C_{x+k+n}+\ldots\ldots),\end{aligned}$$

which, by the preceding formulæ, is

$$\left\{\begin{matrix}aN_{x+k,n}+hS_{x+k+1,n}+AM_{x+k,n}+HR_{x+k+1,n}\\+sD_{x+k}+tN_{x+k\ \ n}+TM_{x+k+n}\end{matrix}\right\}\quad\ldots\quad(A).$$

The balance of the premiums and returns is the following divided by D_x:—

$$\begin{aligned}&(\sigma+\varpi)\mathrm{D}_x+\varpi(\mathrm{D}_{x+1}+\ldots\ldots+\mathrm{D}_{x+l-1})+\varpi\mathrm{D}_{x+l}+(\varpi\overline{1+\mu}+\beta)\mathrm{D}_{x+l+1}\\ &\quad+\ldots\ldots+\{\varpi(1+\overline{m-1}\mu)+\overline{m-1}\beta\}\mathrm{D}_{x+l+m-1}+\tau(\mathrm{D}_{x+l+m}+\ldots\ldots)\\ &\quad-(\rho+\nu\varpi)\mathrm{C}_x-(\rho+\theta+2\nu\varpi)\mathrm{C}_{x+1}-\ldots\ldots-(\rho+\overline{w-1}\theta+w\nu\varpi)\mathrm{C}_{x+w-1},\end{aligned}$$

the value of which, by the same formulæ, is

$$\left\{\begin{matrix}\sigma\mathrm{D}_x+\varpi\mathrm{N}_{x-1,\,l+m}+\varpi\mu\mathrm{S}_{x+l,\,m}+\beta\mathrm{S}_{x+l,\,m}+\tau\mathrm{N}_{x+l+m-1}\\ -\rho\mathrm{M}_{x,\,w}-\theta\mathrm{R}_{x+1,}\ -\nu\varpi\mathrm{R}_{x,\ +1}\end{matrix}\right\}\quad .\quad .\quad \text{(B)}.$$

The equation of the results (A) and (B) gives

$$\left.\begin{matrix}\varpi\{\mathrm{N}_{x-1,\,l+m}+\mu\mathrm{S}_{x+l,\,m}-\nu\mathrm{R}_{x,\,w+1}\}\\ +\sigma\mathrm{D}_x+\beta\mathrm{S}_{x+l,\,m}+\tau\mathrm{N}_{x+l+m-1}\\ -\rho\mathrm{M}_{x,\,w}-\theta\mathrm{R}_{x+1,\,w}\end{matrix}\right\}=\left\{\begin{matrix}s\mathrm{D}_{x+k}\\ +a\mathrm{N}_{x+k,\,n}+h\mathrm{S}_{x+k+1,\,n}\\ +\mathrm{AM}_{x+k,\,n}+\mathrm{HR}_{x+k+1,\,n}\\ +t\mathrm{N}_{x+k+n}+\mathrm{TM}_{x+k+n}.\end{matrix}\right.$$

This problem contains the circumstances of all which are proposed, and is here introduced that any question may have the result of the common investigation compared with that deduced by considering it as a particular case of the preceding. Most of the letters will be $=0$ in any question which occurs; and the following list will serve to remind the calculator what letters enter into the case before him.

1	k	s	Occurs in questions which comprise a fixed sum at a certain age; an endowment.
2	k, n	a	An annuity, present or deferred, &c.
3	k, n	h	An increasing or decreasing annuity.
4	k, n	A	A fixed assurance of any kind.
5	k, n	H	An increasing or decreasing assurance.
6	k, n	t	An increasing or decreasing annuity, with an end to the increase, &c.
7	k, n	T	An increasing or decreasing assurance, with an end to the increase, &c.
8	—	σ	Present value of any kind.
9	l, m, w	ϖ	Fixed premiums of any kind.
10	l, m	μ	Premiums increasing or diminishing by a proportion of the first premium.
11	l, m	β	Premiums increasing or decreasing by a sum independent of the first premium.
12	l, m	τ	Premiums increasing or decreasing with an end to the increase, &c.
13	w	ρ	Return of a sum in case of death.
14	w	θ	Return of increasing or decreasing absolute sum in case of death.
15	w	ν	Return of increasing or decreasing proportions of the first premium in case of death.

The third column contains the letters indicating benefits or payments, and the second column shows the terms of years with which the benefits, &c. are particularly connected in the general problem. By attention to the conditions, the solution of any case can be readily picked out of the general equation, as in the following instances.

What is the premium to be paid for an insurance of £A on a life aged x, accompanied by a return at death of all the premiums paid?

Here all the letters of the third column vanish except A, ϖ, and ν, and m, n, and w are to be extended beyond the possible term of life, while $l=0$, $k=0$, and $\nu=1$. Again, when the age $x+y$ extends beyond the term of life, $N_{x,y}=N_x$, &c. Consequently the equation gives.

$$\varpi(N_{x-1}-R_x)=AM_x.$$

Again, what is the present value of an assurance of £A, with which the sum paid is to be returned? Here only σ, ρ, and A have value, and $\sigma=\rho$, the letters of the first column being as before. Hence.

$$\sigma(D_x-M_x)=AM_x.$$

If D_x be ever less than M_x, this problem is impossible: but D_x is necessarily greater than M_x, being $C_x(1+r)+C_{x+1}(1+r)^2+\ldots$, while M_x is $C_x+C_{x+1}+\ldots$.

Thirdly, an annuity of £a to commence when a life aged x attains $x+k$ is to be bought by a premium regularly diminishing, so as to be last paid when the annuity begins (that is, at $x+k$), and a year before a payment of the annuity is made. Here only ϖ, μ, and a, have value; k is given, $l=0$, m is $k+1$, n outruns the term of life, and $w=0$. And μ must be taken negatively, as $-1\div(k+1)$. Hence

$$\varpi\left(N_{x-1,k+1}-\frac{1}{k+1}S_{x,k+1}\right)=a.N_{x+k},$$

or (VII.) $$\varpi=\frac{aN_{x+k}}{N_{x-1}-(S_x-S_{x+k+1})\div(k+1)}.$$

Before we proceed further, it may increase the interest attached to the formulæ if we remark that the principle of these commutation tables (as we have called them) can be extended from the case of life contingencies to that of interests certain, in such manner that every formula which gives the value of one of the former, may, by going to a different table, be applied to the corresponding one of the latter. That is to say, the mathematical treatment of

the hypothesis that a life is to last for ever does not differ from that of a table of *mortality*. We should imagine, that in questions of instalments particularly, increasing or decreasing, such tables would be of very great use.

To construct them, proceed as follows:—

$$D_x=v^x, \quad N_x=\frac{v^{x+1}}{1-v}, \quad S_x=\frac{v^{x+1}}{(1-v)^2}.$$

The remaining quantities are useless, being always $=0$. $N_{x,y}$ and $S_{x,y}$ may be exhibited as before in the forms

$$N_{x,y}=N_x-N_{x+y}, \quad S_{x,y}=S_x-S_{x+y}-yN_{x+y-1}.$$

Suppose, for instance, we take the last question, and require the value of the annuity for the *whole life* (which here means perpetuity) of a after the expiration of k years, to be bought by regularly diminishing instalments one paid now, &c. The formula then becomes

$$\pi=\frac{aN_k}{N_{-1}-(S_0-S_{k+1})\div\overline{k+1}},$$

and $N_{-1}=\frac{1}{1-v}$, $S_0=\frac{v}{(1-v)^2}$. If the value of the tabular quantities be restored, the preceding (cleared of fractions) is

$$\frac{a(k+1)v^{k+1}(1-v)}{(k+1)(1-v)-(v-v^{k+2})};$$

the same as would be obtained by common methods. The first is (with tables) as easily calculated as the second by the common tables; or, if anything, somewhat more easily.

The particular uses of such commutation tables for certain interests would be—1. That those who can use the life tables more readily than the usual tables of interest (many, perhaps most, actuaries) would at once be able to apply their facility in the former to the new form of the latter. 2. That whenever the formula is given in terms of D_y, N_y, and S_y, it is indifferent at what age the perpetual life of the problem is supposed to begin, so that a repetition of the simple process upon another age verifies the computation.

We now come to the classification of problems, and the presentation of their results. In all cases one of the quantities in the third column is to be unknown, and found from the equation. Two cases arise—1. Where all the quantities of the third column are independent of each other. 2. Where one is to be a simple fraction of another. Thus τ, the premium remaining over, or

what we might call the *residual* premium, might be required to be $\gamma\varpi$, a given fraction of ϖ; and the problem might as easily be solved if all except, say a, (and the indicators of fractions already existing, as μ, ν) were to be made given fractions or multiples of ϖ.

If $(\nu\varpi)$, in parentheses, be taken as an abbreviation of the phrase "coefficient of $\nu\varpi$," &c., we may, for purposes of general consideration, write the equation

$$\varpi\{(\varpi)+(\mu\varpi).\mu-(\nu\varpi).\nu\}+\&c.=(s).s+(a).a+\&c.,$$

the first side entirely depending upon the mode of offering payment, and the second upon the nature and amount of the benefit. Hence it is useless to combine each of the different benefits with all the modes of paying for it; for, as cannot but have been observed by those who have used these tables, a given benefit must be always calculated by the same numerator, whatever the single mode of payment may be; and the payment by the same denominator, whatever the benefit may be. Thus, if a simple deferred annuity be bought by uniform premiums, we have

$$(\varpi).\varpi=(a).a, \quad \text{or } \varpi=\frac{(a)}{(\varpi)}.a.$$

But if the single value be paid for the same, we have

$$(\sigma)\sigma=(a).a, \quad \text{or } \sigma=\frac{(a)}{(\sigma)}.a.$$

But if the mode of payment be double, say partly by a single value, partly by a succession of uniform premiums, we have

$$(\varpi).\varpi+(\sigma)\sigma=(a).a;$$

from which ϖ may be given, and σ be found, or *vice versâ*. If σ is to be a given fraction of ϖ, the *payment* part of the calculation is wholly in the denominator. The following rules will be found useful as preservatives from error:—

1. When no part of the benefit is to depend upon the unknown item of payment,* no function of the benefit can be in the denominator; and the contrary.

2. When all the items of payment are fractions of one among them, no function of the payment can be in the numerator, and the contrary: but when there are parts of the payment not so connected, those which are known are found in the numerator.

3. In all the cases not before specified, the numerator is entirely a function of the benefit and the denominator of the payment.

* With the payment class any returns of payment in case of the conditions of benefit ceasing to exist before it comes due.

If we call the two sides of the equation the payment side and the benefit side, and if, taking twenty different cases of each, we write down the corresponding sides of the equation, we have the materials for solving instantaneously any one out of four hundred problems, out of which all will be practically useful, in which the conditions of the problem make all payments cease at or before the time when the benefit begins. This we proceed to do. The equation is

$$\left.\begin{array}{l} \qquad\sigma D_x - \rho M_{x,w} \\ +\varpi\{N_{x-1,l+m} + \mu S_{x+l,m} - \nu R_{x,w+1}\} \\ +\tau N_{x+l+m-1} + \beta S_{x+l,m} - \theta R_{x+1,w} \end{array}\right\} = \left\{\begin{array}{l} \qquad\qquad sD_{x+k} \\ +aN_{x+k,n} + hS_{x+k+1,n} + tN_{x+k+n} \\ +AM_{x+k,n} + HR_{x+k+1,n} + TM_{x+k+n} \end{array}\right.$$

$$\begin{array}{l|l} N_{x,y} = N_x - N_{x+y} & S_{x,y} = S_x - S_{x+y} - yN_{x+y-1} \\ M_{x,y} = M_x - M_{x+y} & R_{x,y} = R_x - R_{x+y} - yM_{x+y-1} \end{array}$$

The following are the principal cases of benefits to be bought, and under each is written the benefit side of the equation in which it enters. The age of the life is x throughout.

Benefit Terms.—Annuities.

1. *Endowment.—£s* to be received in k years if the party be then alive,

$$sD_{x+k}.$$

2. *Life annuity of £a.*—First payable in one year, continuing through life,

$$aN_x.$$

3. *Deferred life annuity.*—Deferred for k years, makes payment in $k+1$ years,

$$aN_{x+k}.$$

4. *Temporary annuity.* — Makes no payment after n years though the annuitant continue alive,

$$a(N_x - N_{x+n}).$$

5. *Temporary deferred annuity.*—Deferred k, continues n years, first payment in $k+1$ years,

$$a(N_{x+k} - N_{x+k+n}).$$

6_1. *Increasing or decreasing life annuity.*—Differs from (2) in the successive payments being a, $a \pm h$, $a \pm 2h$, &c.,

$$aN_x \pm hS_{x+1}.$$

6_2. When $h = a$, for the *increasing* annuity,*

$$aS_x.$$

* This case can be easily calculated from the common life tables, by a method given by the author of this article in his Essay on Probabilities (Cabinet Cyclopædia).

7_1. *Deferred increasing or decreasing annuity.*—Deferred k years, first payment a, in $k+1$ years; second, $a\pm h$ in $k+2$ years, &c.

$$aN_{x+k}\pm hS_{x+k+1}.$$

7_2. When $h=a$, for the *increasing* annuity,

$$aS_{x+k}.$$

8_1. *Temporary increasing or decreasing annuity.*—Lasts n years only,

$$a(N_x-N_{x+n})\pm h(S_{x+1}-S_{x+n+1}-nN_{x+n}).$$

8_2. When $h=a$, for the *increasing* annuity,

$$a(S_x-S_{x+n}-nN_{x+n}).$$

9_1. *Temporary deferred increasing or decreasing annuity.*—Deferred k years, continues n years,

$$a(N_{x+k}-N_{x+k+n})\pm h(S_{x+k+1}-S_{x+k+n+1}-nN_{x+k+n}).$$

9_2. When $h=a$, for the *increasing* annuity,

$$a(S_{x+k}-S_{x+k+n}-nN_{x+k+n}).$$

10. *Decreasing annuity, temporary by extinction.*—That is, it lasts n years, and each payment is less than the preceding by 1-nth part of $a(hn=a)$,

$$a\left\{N_x-\frac{1}{n}(S_{x+1}-S_{x+n+1})\right\}.$$

11. *Deferred decreasing annuity, temporary by extinction.*—Deferred k years, expires after $k+n$ years,

$$a\left\{N_{x+k}-\frac{1}{n}(S_{x+k+1}-S_{x+k+n+1})\right\}.$$

12. *Arrested increasing or decreasing annuity.*—Here, after n years, when the annuity would begin to pay $a\pm nh$, $a\pm(n+1)h$, &c., the increase or decrease is *arrested*, and it pays $a\pm(n-1)h$ for the rest of life,

$$aN_x\pm h(S_{x+1}-S_{x+n}).$$

13. *Increasing or decreasing annuity, deferred and arrested.*—The period of deferment is k years, and the increase or decrease continues n years; after which, as in the last,

$$aN_{x+k}\pm h(S_{x+k+1}-S_{x+k+n}).$$

14_1. *Temporary annuity, continued by increase or decrease.*—Here the annuity is a for $n+1$ years,* after which it increases or decreases by h for $p-1$ years, and then stops.

$$a(N_x-N_{x+n+p})\pm h(S_{x+n+1}-S_{x+n+p+1}-pN_{x+n+p}).$$

* Namely, for n years from the first part, and one year of the continuation.

14_2. The same *continued to extinction*, $(hp=a)$.

$$a\left\{N_x-\frac{1}{p}(S_{x+n+1}-S_{x+n+p+1})\right\}.$$

15_1. *Deferred temporary annuity, continued by increase or decrease.*—Here, after k years, a is paid for $n+1$ years, and $a\pm h$, $a\pm 2h$, &c., during $p-1$ years more,

$$a(N_{x+k}-N_{x+k+n+p})\pm h(S_{x+k+n+1}-S_{x+k+n+p+1}-pN_{x+k+n+p}).$$

15_2. The same *continued to extinction*, $(hp=a)$.

$$a\left\{N_{x+k}-\frac{1}{p}(S_{x+k+n+1}-S_{x+k+n+p+1})\right\}.$$

In the preceding list it is obvious that any benefit there described is converted into another of the same kind, but deferred for k years, simply by changing x into $x+k$. In like manner the benefit might be anticipated a year, by writing $x-1$ for x, which would make all the immediate annuities become due, or would alter their technical character from *annuities* to *premiums*. Similarly, if we compare the meanings of $D_{x+k}\div D_x$ and $C_{x+k}\div D_x$, we see that

The first is the value of £1 to be received if the person *begin* his $(x+k+1)$th year, whether he live through it or not.	The second is the value of £1 to be received if the person *begin* his $(x+k+1)$th year, *and do not live to finish it.*

If, then, we change D_x into C_x, &c., in any problem of annuities, and alter the benefit side of the equation accordingly, we make a change of benefits as follows:—At every period at which the claimant, being alive, should receive a sum of £1, let him receive it a year later, but only if he die within the year, and let it be forfeited if he live. Consequently an annuity to be paid, say at the seventh, eighth, and ninth birthday from the present time, would thus be turned into an assurance to be paid at the eighth, ninth, or tenth birthday, if the party should die in either of these years. But since M_x was made to begin a year earlier than N_x, or $M_x=C_x+\ldots$ and $N_x=D_{x+1}+\ldots$, this change of conditions as to time is compensated by the structure of the tables; and any one of the preceding annuity benefits is converted into its corresponding assurance benefit, so far as the benefit side of the equation is concerned, by changing N_x into M_x and S_x and R_x. But if D_x ever occur, we must change it into C_{x-1} if the time of payment is to be the same in both.

We might thus dispense with the following list, but, in an

article of reference it is desirable, were it only to avoid the necessity of looking under one head while thinking of another.

Benefit Terms.—Assurances.

1. *Endowment assurance.*—£S to be received in k years, if the person now aged x died in the preceding year,

$$SC_{x+k-1}.$$

2. *Life assurance* of £A.*—Payable at the end of the year of death,

$$AM_x.$$

3. *Deferred assurance.*—Payable at death, if more than k years hence,

$$AM_{x+k}.$$

4. *Temporary assurance.*—Payable at death if within n years,

$$A(M_x - M_{x+n}).$$

5. *Temporary deferred assurance.*—Payable at death, if between the ages of $x+k$ and $x+k+n$,

$$A(M_{x+k} - M_{x+k+n}).$$

6_1. *Increasing or decreasing life assurance.*—Payable at death, A if in the first year, A±H if in the second, A±2H if in the third, &c.,

$$AM_x \pm HR_{x+1}.$$

6_2. When H=A, for the *increasing* assurance,

$$AR_x.$$

7_1. *Deferred increasing or decreasing assurance.*—Deferred k years, A if death in $(k+1)$th year, &c.,

$$AM_{x+k} \pm HR_{x+k+1}.$$

7_2. When H=A, for the *increasing* assurance,

$$AR_{x+k}.$$

8_1. *Temporary increasing or decreasing assurance.*—If death take place in n years,

$$A(M_x - M_{x+n}) \pm H(R_{x+1} - R_{x+n+1} - nM_{x+n}).$$

8_2. When H=A, for the *increasing* assurance,

$$A(R_x - R_{x+n} - nM_{x+n}).$$

9_1. *Temporary deferred increasing or decreasing assurance.*—Deferred k years, continues n years,

$$A(M_{x+k} - M_{x+k+n}) \pm H(R_{x+k+1} - R_{x+k+n+1} - nM_{x+k+n}).$$

* Actuaries say *assurance*, and others *insurance*. The difference may be made useful in remembering (what the courts of law have not yet found out) that a life *assurance* and a fire *insurance* are very different things.

9_2. When $H=A$, for the *increasing* assurance,

$$A(R_{x+k}-R_{x+k+n}-nM_{x+k+n}).$$

10. *Decreasing assurance, temporary by extinction.*—Payable at death, A if in the first year, $(n-1)$-nths of A if in the second, &c.,

$$A\left\{M_x-\frac{1}{n}(R_{x+1}-R_{x+n+1})\right\}.$$

11. *Deferred decreasing assurance, temporary by extinction.*—Payable A, if death take place in the $(k+1)$th year, &c.,

$$A\left\{M_{x+k}-\frac{1}{n}(R_{x+k+1}-R_{x+k+n+1})\right\}.$$

12. *Arrested increasing or decreasing assurance.*—Here, after n years, when the sum payable should be $A\pm nH$, &c., the increase or decrease is arrested, and $A\pm(n-1)H$ is the assurance for the rest of life.

$$AM_x\pm H(R_{x+1}-R_{x+n}).$$

13. *Increasing or decreasing assurance, deferred and arrested.*—Deferment k years, increase or decrease n years, after which as in the last.

$$AM_{x+k}\pm H(R_{x+k+1}-R_{x+k+n}).$$

14_1. *Temporary assurance, continued by increase or decrease.*—Here the assurance is A for $n+1$ years, after which it increases or decreases by H for $p-1$ years, and then stops.

$$A(M_x-M_{x+n+p})\pm H(R_{x+n+1}-R_{x+n+p+1}-pM_{x+n+p}).$$

14_2. The same, *continued to extinction,* $(Hp=A)$,

$$A\left\{M_x-\frac{1}{p}(R_{x+n+1}-R_{x+n+p+1})\right\}.$$

15_1. *Deferred temporary assurance, continued by increase or decrease.*—Here, after k years, A is the assurance for $n+1$ years, and $A\pm H$, $A\pm 2H$, &c., for $p-1$ years more.

$$A(M_{x+k}-M_{x+k+n+p})\pm H(R_{x+k+n+1}-R_{x+k+n+p+1}-pM_{x+k+n+p}).$$

15_2. The same, *continued to extinction,* $(Hp=A)$,

$$A\left\{M_{x+k}-\frac{1}{p}(R_{x+k+n+1}-R_{x+k+n+p+1})\right\}.$$

We now come to the enumeration of the different cases of the payment side of the equation. This we shall divide into two tables, one expressing the terms dependent on the premiums to be paid, the other the returns (where there are any) to be made in the event of no benefits becoming due.

Payment Terms.

1. *Single premium.*—The whole present value of the benefit, σ, paid at once,

$$\sigma D_x.$$

2. *Life premium.*—£ϖ now, and the same at the end of every year during life,

$$\varpi N_{x-1}.$$

3.—*Temporary premium.*—£ϖ now, and $l-1$ more times, l times in all,

$$\varpi(N_{x-1}-N_{x+l-1}).$$

4_1. *Life premium, increasing or decreasing by a proportion.*—£ϖ now, and $(1\pm\mu)\varpi$, $(1\pm2\mu)\varpi$, &c., in succeeding years,

$$\varpi(N_{x-1}\pm\mu S_x).$$

4_2. When $\mu-1$, for the *increasing premium*,

$$\varpi S_{x-1}.$$

5. *Temporary premium, increasing or decreasing by a proportion.* To last only m years, last premium $(1\pm\overline{m-1}\mu)\varpi$,

$$\varpi\{N_{x-1}-N_{x+m-1}\pm\mu(S_x-S_{x+m}-mN_{x+m-1})\}.$$

6. *Premium temporary by extinction.*—Here $m\mu=1$, and the extinction takes place after m premiums,

$$\varpi\left\{N_{x-1}-\frac{1}{m}(S_x-S_{x+m})\right\}.$$

7. *Arrested proportionally increasing or decreasing premium.*—The premiums of m years are $\varpi, \ldots \varpi(1\pm\overline{m-1}\mu)$, at which they afterwards remain,

$$\varpi\{N_{x-1}\pm\mu(S_x-S_{x+m-1})\}.$$

8_1. *Temporary premium, continued by proportional increase or decrease.*—Here $l+1$ premiums ϖ are to be paid: afterwards $m-1$ premiums $\varpi(1\pm\mu)$, $\varpi(1\pm2\mu)\ldots\varpi\{1\pm(m-1)\mu\}$,

$$\varpi\{N_{x-1}-N_{x+l+m-1}\pm\mu(S_{x+l}-S_{x+l+m}-mN_{x+l+m-1})\}.$$

8_2. The same *continued to extinction*, $(\mu m=1)$,

$$\varpi\left\{N_{x-1}-\frac{1}{m}(S_{x+l}-S_{x+l+m})\right\}.$$

9. *Life premium, increasing or diminishing by an absolute sum.*—Premium ϖ, $\varpi\pm\beta$, $\varpi\pm2\beta$, &c.,

$$\varpi N_{x-1}\pm\beta S_x.$$

10. *Temporary premium, increasing or diminishing absolutely.*—To last m years, last premium $\varpi\pm(m-1)\beta$,

$$\varpi(N_{x-1}-N_{x+m-1})\pm\beta(S_x-S_{x+m}-mN_{x+m-1}).$$

11. *Arrested absolutely increasing or diminishing premium.*—The first m premiums are ϖ, $\varpi \pm \beta$, ... $\varpi \pm (m-1)\beta$, at which they afterwards remain,

$$\varpi N_{x-1} \pm \beta(S_x - S_{x+m-1}).$$

12. *Temporary premiums continued by absolute increase or decrease.*—Here $l+1$ premiums ϖ are to be paid; afterwards $m-1$ premiums $\varpi \pm \beta$, $\varpi \pm 2\beta$, ... $\varpi \pm (m-1)\beta$,

$$\varpi(N_{x-1} - N_{x+l+m-1}) \pm \beta(S_{x+l} - S_{x+l+m} - mN_{x+l+m-1}).$$

The following is the table of modes of returning a portion of the premiums.

N.B. These tables do not suffice to calculate the effect of the return of a given proportion of varying premiums. The quantities following are *positive* when put on the benefit side, and negative when on the payment side. It must, of course, be obvious that this is only another table of the values of assurances, described so as to meet the form in which problems are usually given.

Return Terms.

1. *Fixed return at death.*—A fixed sum ρ returned whenever the death may take place,

$$\rho M_x.$$

2. *A fixed return at death, if before w years have elapsed,*

$$\rho(M_x - M_{x+w}).$$

3. *Return at death of a proportion of fixed premiums.*—That is, $\nu\varpi$, if the death take place in the first year, $2\nu\varpi$ if in the second, &c.

$$\nu\varpi R_x.$$

4. *Return at death of a temporary portion of fixed premiums,* if the death take place within w years,

$$\nu\varpi(R_x - R_{x+w} - wM_{x+w}).$$

5. *An increasing or decreasing sum returned at death.*—ρ if in the first year, $\rho \pm \theta$ if in the second, &c.

$$\rho M_x \pm \theta R_{x+1}.$$

6. *The same if the death place in w years,*

$$\rho(M_x - M_{x+w}) \pm \theta(R_{x+1} - R_{x+w+1} - wM_{x+w}).$$

7. *An arrested increasing or decreasing sum returned at death.*—ρ if in the first year, $\rho \pm (w-1)\theta$ if in the wth or any following year,

$$\rho M_x \pm \theta(R_{x+1} - R_{x+w}).$$

8. *A fixed sum, ρ, or a fixed proportion of premiums* returned if the life continue w years,*

$$\rho D_{x+w}.$$

A few general rules will be readily collected from the preceding, and may be simply demonstrated. They might be made the foundation of a synthetical view of the subject.

1. Every thing depends on the fundamental calculation of the various cases of the benefit side of the equation.

2. The benefit side of the equation being found for the whole life, that for the same benefit deferred k years is found by writing $x+k$ for x.

3. And that for the same benefit to last n years is found by changing

$$N_x \text{ into } N_{x,n}, \quad M_x \text{ into } M_{x,n}.$$

With regard to S_x and R_x, the change must be regulated by the following consideration. When their exponent is *one more* than the present age, or the age at a term mentioned in the problem, change S_x into $S_{x,n}$, and R_x into $R_{x,n}$. But whenever S_x or R_x has the exponent of the present age, or of that at the beginning of a term, change S_x into $S_{x,n+1}$ and R_x into $R_{x,n+1}$ (compare (page 16),† the transition from 6_1 to 8_1 with that from 6_2 to 8_2). If no simplifications were allowed—that is, if N or M were always retained for the permanent portion of an annuity or insurance, and S or R for the term depending on the value of the incremental portion, the first rule would be sufficient.

4. All the cases are then derived from the following:—

N_x	on the benefit side of the equation,	an annuity	of £1, £1, £1, &c.
M_x	,,	,, assurance	of £1, £1, £1, &c.
S_x	,,	,, annuity	of £1, £2, £3, &c.
R_x	,,	,, assurance	of £1, £2, £3, &c.

5. Every formula in which any particular relations exist should be carefully looked at with a view to simplification.

If we wish to make a benefit begin k years *earlier*, we write $x-k$ for x. In the case of an immediate annuity this is intelligible enough when $k=1$, but N_{x-k}, when k is greater than 1, is the impossible quantity of this branch of algebra. Its meaning is as follows:—suppose a person has been k years in the enjoyment of a benefit, or of the chance of a benefit, for which he ought to have paid when such enjoyment began; suppose also that, had he died during the k years, the claimant of the payment would have had

* For a given year a proportion of the premiums paid by that time is simply a fixed sum.
† Pages 339 and 340 of this reprint.—ED. *A. M.*

no means of recovering his rights. According to the principles which regulate these transactions, the holder of the unbought benefit ought to pay the claimant not only all arrears with compound interest, if any, but also compensation for the chance of loss which he has run. The value of such compensation is found by writing $x-k$ for x in the value of the benefit as reckoned from the present time. Thus, $N_{30} \div D_{50}$ is what a person now aged 50 should pay for the past and the future, who has been in unbought possession of an annuity of £1 for 20 years; and $C_{30} \div D_{50}$ is what a person aged 50 should now pay for the unbought chance of having formerly received £1, if he had died between 30 and 31.

This last consideration will be particularly important in its application to the commutation tables for interests certain, as we may thus find the value of all arrears, or may solve a case in which partly arrears and partly prospects are to be valued. For instance, a person engaged to pay a decreasing rent for certain tenements, £a at the end of the first year, $a-h$ of the second, &c., and $a-(n-1)h$ at the end of the nth and last: k years elapse during which he pays no rent, and then his affairs pass into the hands of assignees, who are desirous of paying the arrears and buying the remaining term for one sum. Here, at the commencement, the payment side of the equation is σD_x, and the benefit side is $aN_{x,n}-hS_{x+1,n}$: put the last back k years (assuming the age, which is indifferent,* to be k), and we have

$$\sigma = \frac{a(N_0-N_n)-h(S_1-S_{1+n}-nN_n)}{D_k},$$

which is the sum to be demanded of the assignees. [Take notice that this is not the *legal* demand, since the law will not allow compound interest on neglected arrears: it shows the sum necessary to put the creditors in the position they would have had, if they had received all payments when due, and invested them at compound interest.]

It now only remains to show an example of the mode of proproceeding with the registered cases.

An assurance, to commence in k years, and to be £A, $A+H$ .... $A+(n-1)H$ in the following n years, at which last sum it is arrested, together with an annuity of £a, to begin at the same term and to last n years, is to be bought by present payment of a sum σ, and also of a premium which is extinguished after k years, or in $k+1$ payments, on condition that the sum σ shall be returned,

* In the tables for interests certain it will do equally well to put the payment side forward k years.

with simple interest, if the life drop during the k years. Required the first premium ϖ.

Benefit terms. $\begin{cases} \text{The assurance (13), } AM_{x+k}+H(R_{x+k+1}-R_{x+k+n}) & =V \\ \text{The annuity (5), } a(N_{x+k}-N_{x+k+n}) & =W \end{cases}$

Payment terms. $\begin{cases} \text{The fixed sum } \sigma \text{ (1), } \sigma D_x & =X \\ \text{Premium (6), } \varpi\left\{N_{x-1}-\dfrac{1}{k+1}(S_x-S_{x+k+1})\right\} & =Y \end{cases}$

Return terms. $\begin{cases} \text{Return of } \sigma(1+r) \text{ in the first year, \&c.} & \\ \sigma(1+r)(M_x-M_{x+k})+r\sigma(R_{x+1}-R_{x+k+1}-kM_{x+k}) & =Z \\ \text{or } \sigma(M_x-M_{x+k})+r\sigma(R_x-R_{x+k}-kM_{x+k}). & \end{cases}$

As no further simplification suggests itself, each term had better be calculated for the particular case wanted: we have then

$$X+Y\varpi-Z=V+W,\ \varpi=\frac{V+W+Z-X}{Y}.$$

When it becomes necessary to return proportions of decreasing premiums, new tables must be constructed from S_x and R_x, say Z_x and Y_x, so that

$$Z_x=S_x+S_{x+1}+S_{x+2}+\ldots.$$
$$Y_x=R_x+R_{x+1}+R_{x+2}+\ldots.$$

These, divided by D_x, will give the values of the annuity or assurance, £1, 3, 6, ... , or $n\dfrac{n+1}{2}$ in the nth year: and the effect of these tables, combined with the others, will be to give the value of an annuity or assurance which is $a+hn+qn^2$ in the nth year.

In Mr. Barrett's original method, which is still followed by some actuaries, are three columns only, answering to D, N, and S, which, by aid of the first three formulæ VIII., give C, M, and R. The *great principle* of the method, namely, the formation of tables by which deferred, temporary, and increasing benefits are as easily calculated as those for the whole life, belongs to Mr. Barrett as much as the invention and construction of logarithms to Napier. On the other hand, Mr. Griffith Davies, by the alteration presently noted, and the separate exhibition of M and R (he has not given C, which is of little use in practice, though essential to the theory), has increased the utility and extended the power of the method to an extent of which its inventor had not the least idea; and has all the rest of the claim in the matter which is made for Briggs in the adaptation of logarithms to practical use. Nor must it be forgotten that in all probability this most expeditious mode of conducting operations would not have been now in existence but for the sagacity of Mr. Baily, who, as we have seen, saw further into its merits

than the Royal Society. In Mr. Barrett's form there are three columns, A, B, and C; and A_x is not $a_x v^x$, but $a_x(1+r)^{w-x}$, where w is the greatest age any individual can attain. Also B_x is not $A_{x+1}+\ldots$, but $A_x+\ldots$, so that the value of an annuity on a life aged x is $B_{x+1}:A_x$. Again, C_x is $B_x+B_{x+1}+\ldots$ The following comparisons may be useful to those who are habituated to Barrett's original form, remembering that C_x now means Barrett's third column, and not what it has hitherto stood for:—

$$A_x=D_x v^{-w}, \qquad D_x=A_x v^w,$$
$$B_x=N_{x-1} v^{-w}, \qquad N_x=B_{x+1} v^w,$$
$$C_x=S_{x-1} v^{-w}, \qquad S_x=C_{x+1} v^w.$$

Since, then, Barrett's form is that of Mr. G. Davies multiplied by a constant factor, the former are also *commutation* tables, using the life a year older than the given life in the second and third columns.

In conclusion, we may mention that when a whole table is to be calculated, it may happen that it is better to dispense with the assurance columns by means of VIII. Thus in the case of the premium of assurance for a term of years (as noted by Mr. G. Davies) $\dfrac{M_x-M_{x+n}}{N_{x-1}-N_{x+n-1}}$ is not so convenient as $v-\dfrac{N_x-N_{x+n}}{N_{x-1}-N_{x+n-1}}$.

The only works of which the writer is aware, in which the preceding method, whether called by the name of Barrett or Davies, is treated, are the Appendix to Mr. Baily's Treatise on Life Annuities, &c.; the French translation of the same; Mr. G. Davies' work, already cited; a Note in the Appendix to Mr. Babbage's Treatise on Life Assurance; and the treatise on life annuities, &c., by Mr. Jones, now in course of publication in the Library of Useful Knowledge.

A. DE MORGAN.

University College, London,
October 1, 1839.

Since writing the above, it has struck me that it would be more convenient to make the calculations in page 6 [that is, at the commencement of the article] by writing the logarithms of a_x and a_x-a_{x+1}, one above and the other below the logarithm of v^x, than by writing both of the former below the latter.

General Average. *By* RICHARD MORRISON.

IN the month of June, 1862, after the meeting of the second International General Average Congress held in London, a committee was constituted, "for the purpose of establishing one uniform system of general average throughout the mercantile world." The meeting of the council of the National Association for the Promotion of Social Science, held in York in the autumn of 1864, set apart three days for the consideration of this branch of jurisprudence; and the 26th of September and two following days were occupied with the discussion of the various disputed points connected with the subject, under the presidencies of Sir James Wilde and Sir Fitzroy Kelly. The last-named gentleman, in closing the sitting, in the course of his speech gave his opinion as to the course to be pursued in order to give the force of law to the amendments which had been proposed, with the view to promote the uniformity which is so desirable in connection with the adjustment of claims for general average. He considered that "in order to obtain a legislative sanction to the code which had just been completed, it would be advisable to obtain the distinct approval of the leading commercial bodies, particularly the Chambers of Commerce in the great towns; and to obtain, if possible, assurances on the part of the foreign Governments that they would be prepared to adopt the code upon its adoption in this country. . . . . . If possible, the code or rules should be made a Government measure; failing this, it should be entrusted to at least two independent members, one of whom must be a mercantile man, representing a mercantile constituency, and the other a lawyer of eminence; and that it would be desirable to go to work at once, while the public interest was alive to the measure."

Any remarks upon a subject to which so much importance is attached, if they are the means of making known the principles and practice relating to it, cannot fail to be useful; if made for the purpose of diffusing more widely that knowledge amongst mercantile men generally, or among those who are remotely connected with it as a branch of mercantile law. Perhaps in writing on this subject of all others, the somewhat fallacious observation of Sterne holds good: with regard to novelty in literary writing, he says, "new matter is made as apothecaries make new mixtures, by pouring only out of one vessel into another." But we know that the comparison of the various opinions contained in books have frequently given rise to new ideas, and have tended to the

elucidation of many things obscure and doubtful. The advantages, therefore, of the International Congress alluded to—although, perhaps, underrated by many—appear evident in one sense; that from the free and full discussion bestowed upon it, the subject has been well nigh exhausted as far as argument is concerned; and doubtless the propriety of many suggestions has been enforced upon some, whilst others have been confirmed in views which they did not quite admit, owing to the imperfect opinion they had hitherto formed. "Contention of free minds is the origin of progress." One of the best works on marine insurance and average is called by its author a "compilation";* and, indeed, most works on the usages and laws of trade can be little more. Many customs have been handed down from time immemorial, and are not to be found in the written law. It has been shown repeatedly that the law of insurance is founded as much on the practices of mercantile men as upon the marine law, which has been framed at various times in the ordinances and judicial decisions of different countries and states. Insurance and average, like everything dependent on custom and precedent, progress slowly as regards the improvements which take place in the practice of them. There is so much diversity in the cases which arise, and some points are so complicated, that although it does not at present seem possible to work upon a basis of refined principles, the practice may become more uniform by reconciling many difficulties which have stood in the way of it, owing to the want of place and opportunity for that examination by free discussion, which is of so much advantage for the comparison and consolidation of diverse opinions. This the Congress endeavoured to obviate, and the many nations and states represented indicate the universal desire to obtain the results, which, it may be hoped, will eventually be attained.

Although general average forms a prominent part in the works of writers on marine assurance, it is not *necessarily* connected with that subject, a merchant or shipowner being liable for such a charge whether he be insured or not. But if either of these parties should be insured, the insurer, by the implied or stated terms of the policy granted, becomes liable to indemnify the assured for the general average contribution demanded. In considering, therefore, the law of insurance, many questions connected with this branch of maritime and commercial law must come under consideration. The subject is interesting, both in its relation to

* Preface to Arnould, 1st edition.

marine insurance, and of itself, by reason of the equitable principle on which it is founded; because, also, some points connected with it are, in the words of Justice Park, "of all others the most perplexing in the whole law of insurance."

Average has been compared to a trunk or stock, of which insurance is a "large and material branch proceeding from it." Its origin can be traced far beyond the time when insurance commenced.* It is said to have formed a conspicuous part of the laws of Rhodes, which were promulgated nine hundred years before the Christian era, and were subsequently introduced into the Justinian code. When and how insurance first originated is a question difficult to determine. The Romans knew of bottomry and respondentia, which are a species of insurance; but insurance proper owes its origin to more recent times. Some assert that the Jews were the first who instituted it, but of this there is very little evidence. The earliest ordinance now extant on the subject is that of Barcelona. The exact date is uncertain, but it probably came into notice about A.D. 1435. The next was published at Florence, in 1523, where some articles were drawn up as a basis for insurance, which were subsequently amended in 1537. In 1598 a Chamber of Insurance was established at Amsterdam; and in 1755, Magens, a merchant, republished in England an "Essay" on the subject, which he had previously published in German at Hamburgh. The principle of assurance was first applied to navigation and maritime risks generally.

Many have been the definitions of the term "average." In an old book quoted by Stevens more than half a century ago,† it was said to mean "a certain contribution that merchants and others proportionably make towards the losses of such as have their goods cast overboard for the safety of the ship, of the goods, and of the lives of those in the ship, in a tempest; and this contribution seems to be so called, because it is so proportioned after the rate of every man's *average*, or goods carried." Stevens, who was one of the first who wrote at any length on average, gives the simplest and at the same time a more comprehensive definition: "a contribution made by *all* the parties concerned in a sea adventure, to make good a specific loss or expense incurred by *one* of them, for the general benefit. The word average is not to be found in the body of the ordinary form of a policy of insurance; and although it

* "The positive contract of insurance is of many centuries later date than the implied contract of average."—*Grotius.*

† Cowell's *Interpreter.*

has been observed of that document that it is the most informal instrument ever brought before the notice of the public,* the omission does not arise from fault or informality. If we refer to the definitions quoted above, it will be seen that that term does not primarily apply to any contract of insurance, but to a compensatory arrangement or ordinance between the shipowner and merchant or other proprietor of merchandize. As will be shown hereafter, the values of the respective interests, as employed in the apportionment of the amount expended or loss incurred for the general benefit, are stated without reference to anything connected with insurance. The word is to be found in the note, or memorandum as it is called, at the foot of the policy, and is used three times, but in each case it is employed to distinguish particular average (which technically signifies partial loss by deterioration arising from sea damage), as opposed to general average.†

It will appear evident that all parties concerned in the successful issue of a voyage are intimately associated; and in the event of any unforeseen disaster occurring, by which *all* the interests are endangered, if a portion of one of them is sacrificed for the safety of the whole, such would immediately become a lawful subject of general average contribution. The term has its origin from a danger which is general. One of the first subjects comprised in the term "sacrificed for the general benefit" will be jettison. This usually occurs when a vessel is greatly endangered by stress of the elements, and may comprehend either a throwing overboard of a part of the cargo, or of the furniture or stores of the ship. The circumstances may necessitate it when it becomes needful to lighten a vessel in a storm at sea, or to enable her to escape from an enemy, or for the purpose of floating her when she accidentally gets aground. Many have endeavoured to make the act of jettison the test of fundamental principle for the settlement of every case denominated general average; but, like many others, when extended to circumstances and results not contemplated, its application cannot strictly be allowed, especially as the basis of analogy frequently renders complex questions still more difficult of solution.‡

Before a jettison takes place, it is usual to consult with the

* The common marine policy is a translation of the old Italian policy of the 15th century, introduced into England by the Lombards.

† Stevens mentions that the ordinance of Copenhagen and Hamburgh included *general average* under this warranty. At the present time many of the new forms of policy used both at home and abroad specifically mention general average among the risks enumerated, but this is quite a modern introduction.

‡ "The legitimate province of analogy is rather to silence objections than to establish truth."—Leechman's *Logic*, p. 172.

officers or crew, though such a proceeding is not absolutely necessary. We are told that in former times, when merchants took charge of their goods—acting as shipper, supercargo and consignee in one person—foreign ordinances rendered it imperative that their consent should be asked before the merchandize selected was cast overboard; but their refusal was in no way to interfere with the act contemplated, because nautical men were better qualified to judge as to the actual necessity than merchants, however many voyages they might have made. It is customary to note down the circumstances under which the jettison takes place, and likewise the subjects chosen. Great prudence should be exercised in the selection of the goods thrown overboard, and they should be such as would prove likely to relieve the vessel and at the same time occasion as little sacrifice of property as is compatible with the exigency of the case. Thus, when silver bullion was cast overboard in a storm to lighten the vessel, or, as in more recent cases, when claims have been made for brass cannon used for the same purpose, doubts may be reasonably entertained both as to the necessity of such extraordinary sacrifices and the veracity of representations for obtaining restitution. Minute details are necessary to establish cases of such urgency. The master is supposed to exercise his discretion in the event of a jettison, and he is at liberty to select what articles he pleases, and to determine also the quantity that shall be sacrificed. He may, in a time of extreme danger, and where no other course seems open for saving the lives of the crew, throw the whole cargo overboard.

Such a circumstance as a jettison of part of the cargo is of frequent occurrence in the case of vessels trading from ports in Norway and the east coasts of Sweden to ports on the east coast of Great Britain, laden with timber and deals, and carrying deck cargoes, more particularly in the winter months. With regard to the jettison of the deck load as a sacrifice for the general benefit, much has been said both in its favour and against its being allowed as a subject for contribution. Of course, where a deck load is carried, when the ship is in jeopardy it would become the first subject for sacrifice, on account of its situation in the vessel, being the first at hand and its position impeding in many instances the exertions of the seamen. It may be stated at once that hitherto *insurers* have never been held liable for contribution for jettison of deck load, unless there is an express stipulation to the effect that they undertake the risk. A clause is usually introduced into a policy of insurance on deals or timber, worded thus: "On wood

goods *in and over all*, as may be declared and valued." This includes both cargo in the hold and on deck; or it is sometimes written "including risk of deck load."

When the deck load breaks from its lashings and is washed overboard, or in consequence of its having broken adrift it is found expedient to throw it overboard, such loss is not admitted as a legitimate claim for general contribution. It is not sacrificed for the benefit of all parties concerned; in consequence of endangering the safety of the vessel and crew, it becomes absolutely necessary to jettison it. But where the lashings are *deliberately* severed in time of imminent danger, a claim for general *contribution*, as it is called, is admissible according to legal decisions.* Although the law authorises this claim, many writers on the subject consider that in *principle* it ought not to be allowed, even where it is a "custom of trade" to carry a deck cargo. If the deck cargo renders the ship unmanageable, by her being too deeply laden, it becomes the *cause* of the danger; and according to the principles which govern general average, should be for that reason disallowed. The vessel is then, in the words of one writer, "merely relieved of a burden which ought never to have been placed on her"; and this appears a reasonable argument, unless the ships in that specific trade were built in a different manner from ordinary vessels. Every one having the slightest knowledge of maritime affairs must be aware of the necessity of having the decks clear (especially in stormy latitudes), and the ropes, so many of which lie coiled upon the deck, always in such a condition that they can be let go at any moment. But when the decks are incommoded with timber, and the vessels which carry such cargoes being generally of small tonnage, there is less chance of unforeseen and sudden dangers being averted by prompt action. The law, however, states, that "a usage to load ships in a particular trade must be known to both shipowner and shipper, and they must be taken to have entered into the contract with reference to it." (Baily.)

On the other hand it is argued that the law merely interferes to prevent the carriage of deck cargoes between the 1st September and the 1st May, on account of the extreme danger attached to the carriage of such during those months; that there exists no means of ascertaining when that danger ceases; and that whilst such danger is possible, the allowance for any loss from that kind of sacrifice is contrary to the principles which govern general average.

* The question has been revived, as a subject of litigation, so recently as April, 1864.

Nevertheless the matter is far from being satisfactorily settled, unless the rules proposed at the Social Science Meeting shall meet with general approval. Mr. Baily, in his work on general average, observes that it is hardly to be supposed that the correctness of either view of the question will be generally admitted "until some competent tribunal shall set the matter at rest." There is no uniform practice amongst average adjusters with regard to general contribution, or jettison of deck cargo. Some adjust the claim so that only the same description of cargo *under deck* contributes to that thrown overboard. Others state the claim as an ordinary general average. But they are seldom or ever settled according to the terms of the statement, only the jettison being contributed for by policies with the clause "in and over all."

This subject was fully discussed at the Congress before alluded to, and forms one of the sections of the draft of an Act should the force of law be given to the International Average Rules framed and adopted by the various delegates from the Governments and Commercial Associations of Europe and America. The Congress did not discuss principles for the purpose of framing arbitrary rules, but wished rather to lay down such as might tend to promote uniformity of practice, and for the avoidance of many inconveniences which result from the various methods of treatment adopted in many parts of the world. These rules, if universally adopted, would obviate many difficulties, and lead to more justice being done to commercial men whose interests are confided to representatives of whom they sometimes know but little, but who are nevertheless frequently entrusted with great responsibility. In this Bill to be presented for the consideration of the Legislature, it is stated that the "jettison of timber or deals carried on the deck of a ship in pursuance of a general custom of the trade in which the ship is engaged, shall be made good as general average, in like manner as if such cargo had been jettisoned from below deck." It goes on to state that the jettison of any other deck cargo shall not be made good as general average; and that "if the shipment on deck have been made without the consent or sanction of the shipper or owner of such cargo, the loss resulting from its jettison must be made good by the owners of the ship"; but that if the owner placed the goods on deck with the consent of the shipper, then the loss shall fall upon the shipper or owner of the goods.

By another rule of this section of the Bill, it is proposed that cargo jettisoned from the poop or forecastle of a vessel shall be allowed as general average; but jettisons from all other structures

shall not be allowed, as they are considered part of the deck of the vessel.

It has been mentioned that most underwriters specially undertake the risk of claim for general contribution (which applies only to timber deck loads) by inserting the words "in and over all." But the adoption of this rule practically does away with the distinction and renders all such clauses needless, for general contribution merges into general average; and as the insertion of this and similar memoranda had become so common, no alteration is made in the premium demanded, and consequently no increase in the liability of the underwriter, for if he considered his risk much augmented by the comprehensive amendment to the policy he would demand more premium. It is true the risk may be greater, but competition countervails the loss by taking for the same amount what might be an additional hazard; which act, though perhaps not in itself remunerative, may, on the principles on which marine insurance is conducted, be in the long run favourable.*

There have frequently been many questions raised as to whether jettison from houses on deck—such as are so common in American ships—constitute a claim; for, as above stated, if such structures were considered as *under deck*, it would be a subject of general average; and, if a part of the *deck*, the loss might be (if other than wood goods) borne by the shipowner or proprietor of the goods, according to the understanding between them. It is, by the rule quoted, affirmed that all but the poop or raised after-part of the ship, and forecastle or raised forepart of the ship, are considered as the deck. If, then, goods are thrown from the poop or forecastle, the act is equivalent to a jettison from *under* the deck or hold of the vessel; if from houses on deck, it is considered to be a jettison *off* the deck, and, if other than a timber cargo, not claimable. It is, therefore, the custom to insure all cargo in houses or other structures on deck, if not in the two specified, as "on deck," by which proceeding the insurer is at once aware of the extent of his liability. This was a point which, in the words of one of the leading members, it was "highly advisable to settle. So long as it remained in doubt, claims might be drawn up, but there was no certainty that they would be settled by the parties interested."

When the article sacrificed is the immediate cause of the danger, the loss is not allowed as general average. Such goods as jute, cotton and hemp, are liable when sea damaged to ignite through

* We have good authority for saying that the prudent underwriter does not admit the clause "in and over all" without charging an extra premium.—ED. *A. M.*

spontaneous combustion; and many vessels have, in consequence of such an occurrence, been totally destroyed. When, therefore, any merchandize in that condition is cast overboard, it cannot be claimed as a *sacrifice*. In the like manner spars and booms, water and harness (or salt provision) casks, hawsers and other things which have broken adrift and are rolling about the decks; or boats carried from the chocks or davits; when these are jettisoned because they hinder the proper navigation of the vessel or endanger the lives of the crew, being in a "state of wreck," are equally disallowed. "When property is destroyed because it endangers other property, the owner of the property destroyed is not entitled to compensation from the destroyer."

Water casks on deck are never allowed, as it is said the deck is an improper place for them. Hawsers and chains are generally disallowed for the same reason; but when, in extreme danger, the latter are cast overboard when a vessel is nearing the shore and it is deemed prudent to have them ready, or when a vessel is approaching her port of destination, they are admitted as proper sources of contribution; but under all other circumstances these articles are rejected.

The freight being lost when the merchandize is jettisoned is recoverable; it is contained, as it were, in the goods, for had they been delivered at their destination the freight would have been earned; and being inseparable from the goods is sacrificed for the benefit of all concerned.

More attention has been devoted in this article to the subject of jettison than will be bestowed upon other branches of the subject, on account of its importance—of the many questions involved in it—of its being one of the earliest and one of the most frequent instances of a general average act—and because many have endeavoured to make it the type of all similar proceedings, although it cannot always be brought to bear strictly as a *principle* upon other acts called by that name. Sir J. Arnould says of it that "it is the most perfect example of a general average loss."

It sometimes happens that when a ship is labouring in a heavy sea, during a violent gale of wind, it is found necessary to relieve her by cutting away her sails or masts to avoid a greater danger. Now it would appear at first sight that such a sacrifice would be an undisputed subject of general average, for no prudent seaman would willingly destroy the most important part of his ship's furniture. If a vessel's masts are cut away she is like a helpless log upon the waves, and is liable to receive considerable damage in her

hull and rudder, owing to the want of sails to steady her. But there is no sacrifice which requires more consideration on the part of the captain at the time, and careful evidence as to its propriety when the claim for restitution is subsequently made. Nothing but the most imminent danger will justify the necessity and entitle him to recover a loss of this nature; and that part of the vessel which is sacrificed must be *primâ facie* the cause of relieving the ship from her perilous position. Nevertheless, upon perusing the protests of many foreign masters of vessels, it would almost appear that they are perfectly reckless with the ship's furniture. A sail is split, it is immediately cut adrift; a mast or yard is sprung, without attempting to "fish" it, as is commonly done, it is cut away with the ropes and gear attached; should it injure the bulwarks or boats in falling, all the loss and damage is claimed as general average; and great surprise is manifested when, if made up in this country, the greater part of these sacrifices are excluded as inadmissible under that head. Nor is this merely a modern grievance. Upwards of fifty years ago it was said of this class—

> "Foreigners appear to expect that every mast that is sprung, or sail that is split, when on a lee shore, should be made good by a general contribution; and more particularly if they are afterwards obliged to cut them away. . . . . They should be aware that it is not merely the making use of the axe or knife on the mast, ropes, or sails of the ship, which will constitute a claim for general average."*

When, however, a ship is on her beam ends, and in order to righten her it becomes necessary to cut away masts and spars, or to prevent her from driving on a lee shore, they are admitted. The true test is, whether the voluntary loss will be *supposed* to have the *immediate* effect of saving the remainder of the property in jeopardy. Although the desired and expected results may not be at once apparent, the act must be one of judgment, one calculated to bring about the effect which may be naturally and ordinarily looked for. The furniture of the ship, which, in consequence of the severity of the weather, is in a "state of wreck," is not admissible; the articles must not be those which would be lost inevitably. Sails and rigging blown away and hanging over the ship's side, necessarily drag in the wake of the vessel, and, becoming entangled in the rudder, might occasion the total loss of ship and cargo.† The principle of

* Stevens, p. 17; 2nd edit.

† It will be remembered, that in the hurricane which occurred in the Black Sea on the 14th November, 1854, when, in two hours, eleven of our transport fleet were wrecked and totally lost, and six dismasted off Balaclava, several of them, including the S. S. "Prince," were lost by the entanglement of the wreck of their masts in their rudders, which caused them to lie exposed to the fury of the wind without any possibility of escape.

their being the immediate cause of the danger excludes them from general average.

Another sacrifice which is allowed is the cutting away of the bulwarks when the decks are filled with water. It is seldom that such a case occurs; the weight of the water sometimes carries the bulwarks away, but mariners would not, except as a last resource, *sacrifice* them, a great emergency only would cause them to deprive themselves of the protection afforded by them when the ship is almost unmanageable. When the bulwarks are injured by the masts which have been cut away falling on them, such damage is allowed, because it is the result of an act of judgment and necessity, and belongs to it as such. So also with regard to a like injury to boats.

It is also an unusual occurrence to sacrifice boats; it will be obvious that it would happen only when other means of escape were found unsuccessful. The stern boat is always excluded from general average, on the same principle as the jettison of water casks, viz., that the situation is an improper one.

One other voluntary loss for the general benefit is of the greatest importance—namely, the sacrifice of anchors and chains. This is one of the most frequent causes of contribution. It is done by slipping from them. Anchors are never *cut away* unless they have broken from their lashings in a storm, and are knocking holes in the vessel's bows, in which case the loss falls upon the shipowner. When a vessel is on a lee shore, with a strong breeze blowing and freshening fast to a gale, it may be that time will not allow them to heave up the anchor, so that they might get away in safety; or, when a fleet of outward bound vessels, taking advantage of a sudden change of wind, bear down *en masse*—as frequently happens in the Downs or at the Sandheads off the mouth of the Hooghly—and it becomes needful, in order to avoid a collision, to slip from the anchors and chains: in both these instances a claim for restitution may be made upon the collective interests. All other cases, such as foul anchorage, or chains parting under ordinary circumstances, are disallowed.

Other sacrifices of ship's furniture or materials, as rope cut up to secure masts or to rig temporary masts; sails cut and altered, to be substituted for those lost; or the employment of any materials not originally intended for the use to which they are applied, are allowed as claimable; because it is supposed that a loss of the vessel and the property in her may be averted by such an act, and the extraordinary use of those articles becomes a proper subject for general average.

In stating a claim for the loss of these materials, it is usual, unless the ship be new, to deduct one-third from the cost of replacing them—that is, from the sound value. Although this is an arbitrary rule, it is found to be the only really practical one. The deduction is made for the wear and tear of the things sacrificed. It is true the ship may have had them fitted new but a few weeks or days previous to the emergency which called for their destruction, and they could not have become deteriorated one-third of their value; but, on the other hand, it might happen that when they were destroyed at another time, the period they were originally intended to serve had almost elapsed, and their value might be but a sixth or a tenth of the sound value of replacing them. It would be almost impossible, in a single instance, to determine the point; so that to establish a custom like the one in vogue, does justice, as far as is practicable, to all parties concerned. With regard to chain cables, one-sixth is deducted, "new for old," as it is called; and anchors are allowed in full.

Having referred to the leading cases comprehended in the term *sacrifice*, for which restitution is made, we now turn to the other great principle or source of general average, that of *expenses*; which, when incurred, constitute a claim for *recompense*. In the former case, sacrifices made under circumstances of imminent danger; in the latter, disbursements made for the general benefit. Both necessitate repayment as compensation.

As in treating of sacrifice some subsidiary circumstances were omitted, as not of sufficient importance to be introduced in an article of this character, so in the observations on general average expenses the minor causes of such will either be generalized or remain unnoticed. The vexed questions pertaining to general average, which were so keenly debated at York, in 1864, will then be mentioned—questions which have been controverted for many years, and which cause disputes and additional labour where practices differ. It will then remain to consider that part of the subject which relates to the value of the interests which should contribute: the values of the ship, freight, and cargo, which should be taken in apportioning the amount due for the sacrifice of the goods of the merchant; or for the loss of the like nature incurred by the shipowner, or for expenses incurred by him for the benefit of all interested, or which fall upon the other interests as special charges. Then to add a few remarks upon the method adopted, both here and abroad, in stating average claims belonging more particularly to the province of the average-adjuster.

(*To be continued.*)

NOTICES OF NEW WORKS.

Table of the Reciprocals of Numbers, from 1 *to* 100,000, *with their Differences, by which the Reciprocals of Numbers may be obtained up to* 10,000,000. *By* LIEUT.-COL. W. H. OAKES, A.I.A. London: C. & E. Layton. 1865.

The following observations appear in No. 302 of the *Philosophical Magazine*, and as they very accurately describe Colonel Oakes' work, and indicate its merits, we are glad to adopt them in preference to any remarks of our own on the subject.

"In most, if not all tables of reciprocals of numbers," says the writer, "the arrangement hitherto adopted has been to give integral numbers from 1 to n, and the corresponding reciprocals as decimals. In the present table both the numbers and their reciprocals are given without decimal points, which are to be supplied according to circumstances. Advantage is thus taken of the fact that the significant digits of the reciprocals of, for example, 3·7256, 37·256, 372·56, &c., are 2684131; the reciprocals being respectively 0·2684131, 0·02684131, 0·002684131, &c. In other words, Colonel Oakes has arranged his table with a view to the fact that if r is the reciprocal of n, then will $r\times 10^r$ be the reciprocal of $n\div 10^r$. In consequence he has been able to render the arrangement of the table almost identical with that of an ordinary table of logarithms. In fact the only exception is that the differences reckoned in order of the numbers are all negative; as they must obviously be, since, the numbers increasing, their reciprocals will decrease.

"Prefixed to the table are two notices, one describing the arrangement of the table and exemplifying its uses, the other giving a short account of the manner in which the reciprocals of the larger numbers were calculated. A word or two may be said on the latter point.

Let n denote any large number, K the sum of the arithmetical complements of the logarithms of n and $\overline{n+1}$, or

$$K=\log\frac{1}{n.(n+1)}.$$

Also let d_1, d_2, d_3... denote respectively

$$\log(n+1)-\log(n-1),\ \log n-\log(n-2),\ \log(n-1)-\log(n-3),\ \&c.$$

Now
$$\frac{1}{n-1}-\frac{1}{n}=\frac{1}{n.(n+1)}=\frac{1}{n(n+1)}\cdot\frac{n+1}{n-1}.$$

Therefore

$$\log\left(\frac{1}{n-1}-\frac{1}{n}\right)=\log\frac{1}{n(n+1)}+\log(n+1)-\log(n-1)=K+d_1.$$

Similarly

$$\log\left(\frac{1}{n-2}-\frac{1}{n-1}\right)=K+d_1+d_2,$$

$$\log\left(\frac{1}{n-3}-\frac{1}{n-2}\right)=K+d_1+d_2+d_3,$$

and so on. Of course d_1, d_2, d_3... are given by a table of logarithms,

and are equal for a considerable series of numbers. Hence it is plain that the logarithms of the differences of the successive reciprocals can be obtained by addition, and the calculation conducted in a tabular form. Suppose n to equal 62500, then will K equal 408 2330, and the calculation will stand thus:—

Numbers.		Logs. of Diff. of Reciprocals.	Diff. of Reciprocals.	Reciprocals.
	K =	408 2330		
62500		..	..	·00001600 0000000
	Diff. of logs.			
62499	139	408 2469	256004	0256004
98	139	2608	12	0512016
97	139	2747	20	0768036
96	139	2886	29	1024065
95	139	3025	27	1280102

"The reciprocals entered in the table are, of course,

1600 000, 1600 026, 1600 051, 1600 077,
1600 102, 1600 128, &c.

"In the same tabular calculation the reciprocals of the half and quarter numbers are found by simple multiplication.

"The chief merit of a work of this kind is, of course, accuracy. To secure this, every precaution seems to have been taken. 'To prevent error,' says Colonel Oakes, 'the co-logarithms were checked independently at each 50th term. In taking out the numbers, the progression of their differences was kept in view, so that no material error could occur. The summation of the differences was checked at every 10th term by a subordinate summation, and by comparison with Barlow's tables; and wherever the seventh figure could be uncertain, it was determined by actual division. Finally, every hundredth term was computed by division. The whole of the calculations were performed in duplicate, and when the proofs were set up from one manuscript they were read with the other; and second and third proofs were also each examined by addition of the printed differences, and by comparison with Barlow's table at each 10th term. Lastly, the proofs were again examined, and the whole table virtually recomputed by summation on the Arithmomètre of M. Thomas de Colmar.'

"It is proper to add that the work was undertaken at the suggestion of Professor De Morgan, who says that it is, as far as he knows, 'the largest which has ever been attempted,' and that 'it is a very useful table, and that its applications are far too little known and thought of.'"

CORRESPONDENCE.

ON MR. STEPHENSON'S THEORY OF OPTIONS.

To the Editor of the Assurance Magazine.

SIR,—As I think it possible that your readers may have had nearly enough of this subject, I will be as brief as I can in my reply to the two points raised in Mr. Stephenson's letter in the last Number of the *Journal.*

Mr. S. asserts that I have altered the conditions of his problem. This I deny. His formula is deduced upon the supposition that P_x is deposited at interest; and, therefore, *at the moment of death* there will be due to the depositor's estate P_x plus the interest accruing from the commencement of the year of death. But this is precisely the same thing as $P_x(1+i)$, or P_x with a year's interest, payable at the end of the year of death; which, in accordance with the usual practice, I assumed in my problem.

Mr. S. further states, that in my remark respecting the "value of the policy" the possibility of deteriorated health is left out of consideration. Quite true; it *is* left entirely out of consideration, along with everything else which has nothing to do with the question I have raised. Any reader who may have taken the trouble to follow my reasoning will have noted that I mention the "value of the policy" merely to account for the *possibility* of effecting the assurance without parting with the control over the premium, and will not need any further explanation of the sense in which the expression is used.

If, instead of stating his problem as he has—viz., "to find the premium" required for the assurance, with the option of withdrawal—Mr. Stephenson had simply undertaken *to show how the assurance might be effected* so as to reserve to the policyholder the control over the premium paid, he would have avoided laying himself open to the exception I have taken. His mode of stating the problem, however, showed clearly that his notions on the subject were radically wrong; that he supposed the option of withdrawal to be a benefit included, and charged for, in the premium; and if any further proof of this were required, it is amply supplied in his last letter.

One more shot—not at Mr. Stephenson (who is a stranger to me) but at his theory—and I shall trouble neither your readers nor myself any further with the matter. He says that the value of the annuity, with P_x returnable *at or before death* is $\frac{N_{x+n}}{(N_x-N_{x+n})i+D_{x+n}}$. This expression I have shown is the value of the annuity, with P_x, and a year's interest on it, returnable at the end of the year of death. But P_x with a year's interest at the end of the year of death, is equivalent to P_x with half a year's interest at the moment of death; so that, according to Mr. Stephenson,

1. A deferred annuity,
2. The return of the premium at death, and
3. The option of previous withdrawal,

are together equal in value to

1. A deferred annuity,
2. The return of the premium at death, and
3. Half a year's interest on the premium, at death.

Here, then, we have it at last. The option of withdrawing the premium any time *before death* (and before the expiration of n years), is equivalent to half a year's interest upon it payable *at death* within the same period!

I remain, Sir,

Your very obedient servant,

London, 1*st November*, 1865. W. M. MAKEHAM.

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END OF VOL. XII.

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